Generalized More-Sum-Than Difference Sets

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Motivation

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Natural question: What are the sizes of the sum/difference sets?

Definitions

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

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Definition

Difference dominated: $|A - A| > |A + A|$
Balanced: $|A - A| = |A + A|$
Sum dominated (or MSTD): $|A + A| > |A - A|$.

History

Nathanson, *Problems in Additive Number Theory*. "With the right way of counting the vast majority of sets satisfy $|A - A| > |A + A|$."

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$x + y = y + x$, but $x - y \neq y - x$.

History

Theorem (Martin-O'Bryant): If each set $A \subseteq [0, n-1]$ is equally likely, then a positive percentage of sets are sum-dominant in the limit. More precisely:

$$\lim_{n\to\infty} \frac{\#\{A \subseteq [0, n-1];\ A \text{ is sum-dominant}\}}{2^n} \approx 0.00045.$$

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The trick is to control the fringes.

Notation

- As adding sets and not multiplying, set

$$kA \;=\; \underbrace{A + \cdots + A}_{\text{k times}}.$$

- $[a, b] \;=\; \{a, a+1, \dots, b\}$.

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$?
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$?
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k?

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$? Yes.
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? Yes.
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k? No. (No such set exists.)

Initial Observations

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If A is symmetric ($A = c - A$ for some c) then

$$|A + A| = |A + (c - A)| = |A - A|.$$

$|2A + 2A| > |2A - 2A|$

Example: $|2A + 2A| > |2A - 2A|$

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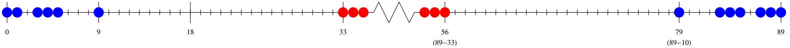
$$A = \{0,1,3,4,5,9\} \cup [33,56] \cup \{79,83,84,85,87,88,89\}$$

$|2A + 2A| > |2A - 2A|$

A



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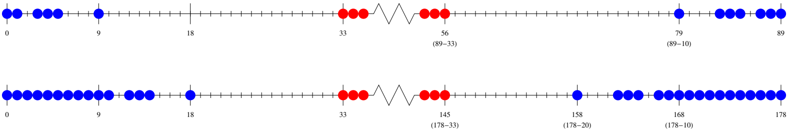
$A + A$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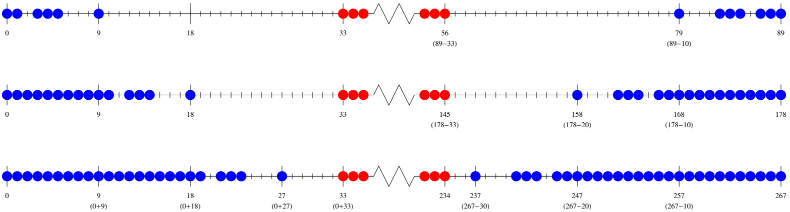

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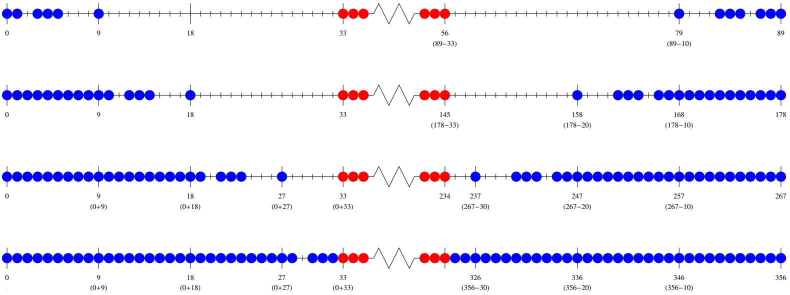

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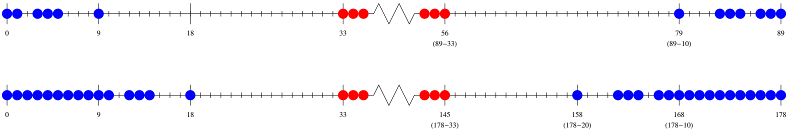
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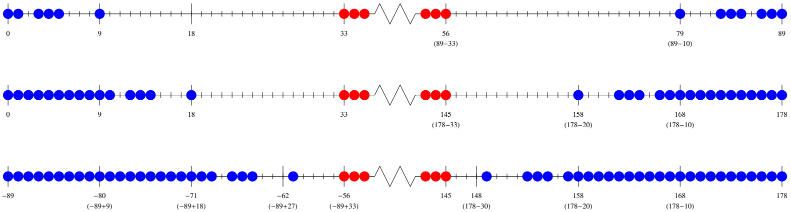
$|2A + 2A| > |2A - 2A|$

$A + A - A$



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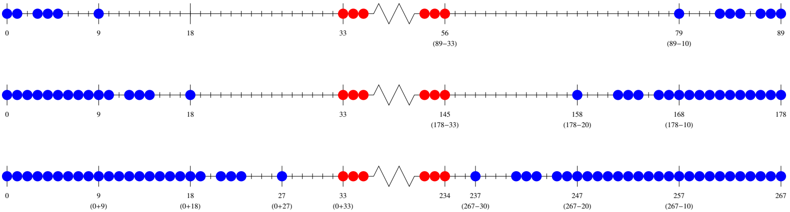

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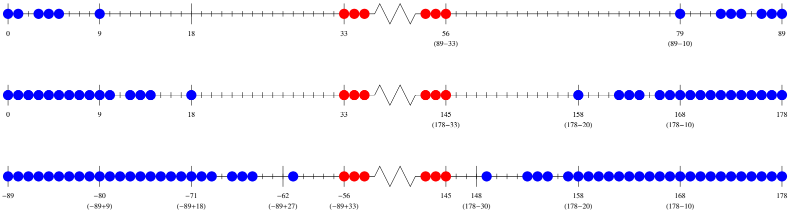

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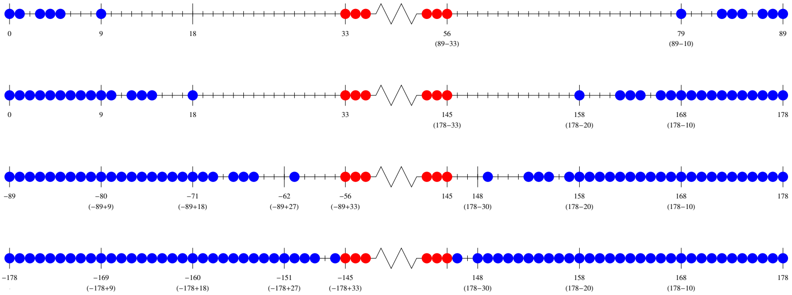

Generalization

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For all nontrivial choices of s_1, d_1, s_2, d_2, $\exists\, A \subseteq \mathbb{Z}$ such that $|s_1A - d_1A| > |s_2A - d_2A|$.

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Example: We can have $|A+A+A+A| > |A+A+A-A|$:

$$A = \{0, 1, 3, 4, 5, 9, 33, 34, 35, 50, 54, 55, 56, 58, 59, 60\}$$

k-Generational Sets

Question: Does a set A exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$?

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Say A is k-generational if $A, 2A, \dots, kA$ all sum-dominant.

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Yes!

$$A = \{0, 1, 3, 4, 7, 26, 27, 29, 30, 33, 37, 38, 40, 41, 42, 43, 46, 49, 50, 52, 53, 54, 72, 75, 76, 78, 79, 80\}$$

In fact, we can find a k-generational set for all k.

k-Generational Sets

Idea of proof: We can find A_j such that $|jA_j + jA_j| > |jA_j - jA_j|$ for a specific $1 \leq j \leq k$.

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Combine the A_j using the method of base expansion.

Base Expansion

Base Expansion: For sets A_1, A_2 and $m \in \mathbb{N}$ sufficiently large (relative to A_1, A_2) the set

$$A = m \cdot A_1 + A_2$$

has the property that

$$|xA - yA| = |xA_1 - yA_1| \cdot |xA_2 - yA_2|$$

whenever $x + y$ is small relative to m.

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whenever $x + y$ is small relative to m.

(here multiplication is the usual scalar multiplication)

Generalization

For nontrivial x_j, y_j, w_j, z_j $(2 \leq j \leq k)$, we can find an A such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all j.

Generalization

For nontrivial x_j, y_j, w_j, z_j $(2 \leq j \leq k)$, we can find an A such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all j.

Example: We can find an A such that

$$|A + A| > |A - A|$$
$$|A + A - A| > |A + A + A|$$
$$|5A - 2A| > |A - 6A|$$
$$\vdots$$
$$|1870A - 141A| > |1817A - 194A|$$

Limiting behavior of kA

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It turns out that all sets have a sort of limiting behavior.

Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

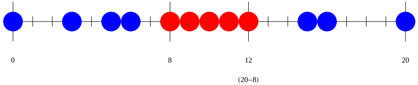

Figure: A

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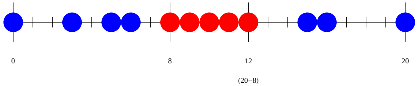

Figure: A

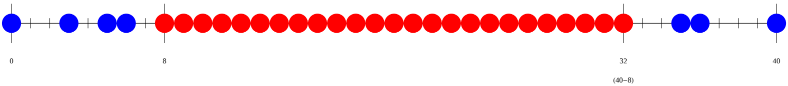

Figure: $A + A$

$|kA - kA|$ vs. $|kA + kA|$

Nathanson: For any set A, kA becomes stabilized before k reaches $\max(A)^2 \cdot |A|$.

We improve this bound to $\max(A)$.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, kA will become difference-dominated or balanced before k reaches $2 \cdot \max(A)$.

Proof Idea:

- Subtraction allows the two fringes to interact, while addition keeps them apart.
- After the set stabilizes, the difference set will "contain" the sum set.

Thanks

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