Low-lying zeros of cuspidal Maass forms

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Introduction

L-functions

Riemann zeta function:

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_{p \text{ primes}} \frac{1}{1 - p^{-s}}$$

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Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Measures of Spacings: n-Level Density

n-level density for one function

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

- Test function $\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i is even Schwartz function.
- Fourier Transforms $\widehat{\phi}$ has compact support: $(-\sigma, \sigma)$.
- Zeros scaled by L_f.
- Most of contribution is from low zeros.

Katz-Sarnak Conjecture

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Need to average *n*-level density over a family and take the limit of this parameter; as $|N| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) \ \to \ \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

Cuspidal Maass Forms

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1-s) > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

L-function associated to Maass forms

Write Fourier expansion of Maass form u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi |n| y) e^{2\pi i n x}.$$

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Define L-function attached to u_j as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$

n-level over a family

- Recall for Katz-Sarnak Conjecture,

$$\begin{aligned}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(L_f\gamma_E^{(j_i)}\right)\\ &\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.\end{aligned}$$

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$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

- For Dirichlet/cuspidal newform L-functions, there are many with a given conductor.
- Problem: For Maass forms, expect at most one with a given conductor.

n-level over a family, continued

- Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

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$$h_{2,T}(t_j) = \exp\left(-(t_j - T)^2/L^2\right) + \exp\left(-(t_j + T)^2/L^2\right),$$

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- Weighted 1-level density becomes

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n,u_j}(\phi)$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\gamma}{2\pi} \log R\right)$$

Results

1-Level Density

1-level density for one function

$$D(u_j;\phi) \quad = \quad \sum_{\gamma} \phi\left(\frac{\gamma}{2\pi}\log R\right)$$

1-Level Density

1-level density for one function

$$D(u_j;\phi)$$

$$= \text{Terms involving } \Gamma + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

$$-\sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

$$+O\left(\frac{1}{\log R}\right)$$

1. Explicit formula.

1-Level Density

1-level density for one function

$$
\begin{aligned}
&D(u_j;\phi)\\
&= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right)\\
&-\sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)\\
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1. Explicit formula.
2. Gamma function identities

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&D(u_j;\phi)\\
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&- \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)
\end{aligned}
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1. Explicit formula.
2. Gamma function identities
3. Prime Number Theorem

Average 1-level density

The weighted 1-level density becomes:

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j;\phi)$$

$$= \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right) + \frac{1}{\sum_j \frac{h_t(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \widehat{\phi}(0) \frac{\log(1+t_j^2)}{\log R}$$

$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

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Kuznetsov Trace Formula

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$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m)\overline{\lambda_j(n)}$$

$= $ some function that depends just on h, m, and n

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where

$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

$$S(n,m;c) = \sum_{0\leq x\leq c-1, gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

$$J_{ir}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+ir+1)}\left(\frac{1}{2}x\right)^{2m+ir}.$$

Kuznetsov Formula

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Result: 1-level density

Theorem (AILMZ, 2011)

If $h_T = h_{2,T}$, $T \to \infty$, and $\sigma < 2/3$ then 1-level density is

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j;\phi) = \frac{\phi(0)}{2} + \widehat{\phi}(0) + O\left(\frac{\log\log R}{\log R}\right) + O(T^{\sigma(3/2+\epsilon)-\eta}).$$

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- This matches with the orthogonal family density as predicted by Katz-Sarnak.

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Can distinguish unitary and symplectic from the 3 orthogonal groups, but 1-level density cannot distinguish the orthogonal groups from each other if support in $(-1, 1)$.

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2-level density can distinguish orthogonal groups with arbitrarily small support; additional term depending on distribution of signs of functional equations.

Result: 2-level density

Theorem (AILMZ, 2011)

Same conditions as before, for $\sigma < 1/3$ have

$$
\begin{aligned}
D^*_{2,\mathcal{F}} \;=\; & \prod_{i=1}^{2}\left[\frac{\phi_i(0)}{2} + \widehat{\phi}_i(0)\right] + \frac{1}{2}\int_{-\infty}^{\infty} |z|\widehat{\phi}_1(z)\widehat{\phi}_2(z)dz \\
& -\phi_1(0)\phi_1(0) - 2\widehat{\phi_1\phi_2}(0) + (\phi_1\phi_2)(0)\mathcal{N}(-1) \\
& +O\left(\frac{\log\log R}{\log R}\right).
\end{aligned}
$$

Note that $\mathcal{N}(-1)$ is the weighted percent that have odd sign in functional equation.

Conclusion

Recap

- We calculated 1-level for $\sigma < 2/3$.
- Calculated 2-level densities for $\sigma < 1/3$ in order to distinguish the orthogonal families.
- We showed agreement with Katz-Sarnak conjecture.

Thank you!