n-Level Densities of Zeroes of Quadratic Dirichlet L-Functions

Jake Levinson
Advisor: Steven J. Miller (Williams College)

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Random Matrices

- Physics: eigenvalues $\leftrightarrow$ energy levels of complex systems
- Spacings, distributions, clusters, ...

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G(N)	**Matrix Ensemble**
$U(N)$	Unitary $N \times N$ matrices
$SO(N)$	Unitary orthogonal $N \times N$ matrices
$USp(N)$	Unitary symplectic $N \times N$ matrices

- $G(N)$ is a probability space
  - can integrate (Haar measure) and study statistics
  - Unitary matrix: eigenvalues $e^{i\theta}$ on unit circle

Analytic Number Theory

Studies questions like:

- How many prime numbers in $\{1, \ldots, n\}$?
- How large are the spacings between prime numbers?
- How common are clusters of primes?

Techniques used for these questions can be used to study primes in arithmetic progressions, elliptic curves, number fields, ...

L-functions

An L-function $L(s, f)$ is defined by a series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p L_p(s, f).$$

(Riemann zeta: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$.)

- **Explicit Formula** relating quantity or object of interest to a sum over the zeroes of L
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  (Riemann zeta: counting the primes up to n)
- **Functional Equation** $L(1-s,f) = g(s)L(s,f)$
  - Error term for quantity of interest is smallest if the zeros have real part $\Re(s) = \frac{1}{2}$.

Symmetry and zeros

- **Generalized Riemann Hypothesis (GRH)**: For 'nice' L-functions, all the zeroes with $0 < \Re(s) < 1$ in fact have real part $\Re(s) = \frac{1}{2}$.

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- **Spacing statistics**: Label the zeroes of $L(s, f)$ as $\frac{1}{2} + i\gamma_j$,

$$\cdots \leq \gamma_{-2} \leq \gamma_{-1} < 0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$$

  and study the statistics of the γ_i.
  - L-functions zero spacings $\leftrightarrow$ RMT eigenvalue spacings

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- L-function family: $\mathcal{F}$ = quadratic Dirichlet L-functions
- Problem: need to average over infinitely-many L-functions!
  - 'parametrize' by conductor to get finite subset $\mathcal{F}(X) \subset \mathcal{F}$; then let $X \to \infty$

n-level Densities - Number Theory

Let $f_1, \dots, f_n$ be even Schwartz functions. For a fixed $L \in \mathcal{F}$ we let

$$D(L; f_1, \dots, f_n) = \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} f_1(\tilde{\gamma}_{j_1}) \cdots f_n(\tilde{\gamma}_{j_n}).$$

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We **average over the family**:

$$D(\mathcal{F}, X; f) = \frac{1}{|\mathcal{F}(X)|} \sum_{L \in \mathcal{F}(X)} D(L; f_1, \dots, f_n).$$

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We study the limit

$$\lim_{X \to \infty} D(\mathcal{F}, X; f) = \int_{\mathbb{R}^n} f_1(x_1) \cdots f_n(x_n) W_{\mathcal{F}}^{(n)}(\mathbf{x}) d\mathbf{x}.$$

n-level Densities - Random Matrix Theory

Let $f_1, \dots, f_n$ be even Schwartz functions. For a fixed $A \in G(N)$ we let

$$D(A; f_1, \dots, f_n) = \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} f_1(\tilde{\theta}_{j_1}) \cdots f_n(\tilde{\theta}_{j_n}).$$

We **average over** G(N):

$$D(G(N); f) = \int_{G(N)} D(A; f_1, \dots, f_n) dA.$$

We study the limit

$$\lim_{N \to \infty} D(G(N); f) = \int_{\mathbb{R}^n_{\geq 0}} f_1(x_1) \cdots f_n(x_n) W_G^{(n)}(\mathbf{x}) d\mathbf{x}.$$

The Katz-Sarnak Density Conjecture

- We associate one of the matrix groups G to our family of L-functions
- Katz-Sarnak Density Conjecture:

$$\underbrace{\lim_{X\to\infty} D(\mathcal{F},X;f)}_{\text{density of zeros}} = \underbrace{\int_{\mathbb{R}^n} f(\mathbf{x})W_G^{(n)}(\mathbf{x})d\mathbf{x}}_{\text{eigenvalue density}}.$$

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(Quadratic Dirichlet L-functions: $G = \mathbf{USp}$ (symplectic).)

Thesis Results

Previous results:

Let $f_1, \ldots, f_n$ be even Schwartz functions, and consider the density conjecture

$$\lim_{X \to \infty} D(\mathcal{F}(X); f_1, \ldots, f_n) = \int_{R^n_{\geq 0}} f_1(x_1) \cdots f_n(x_n) W_{USp}^{(n)}(\mathbf{x}) d\mathbf{x}. \quad (*)$$

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  If $\hat{f}_1, \dots, \hat{f}_n$ are supported in $\sum_{i=1}^n |u_i| < 1$, equation $(*)$ holds.

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  If $\hat{f}_1, \dots, \hat{f}_n$ are supported in $\sum_{i=1}^{n} |u_i| < 2$, we can compute both sides of $(*)$ for all n; equality holds for $n = 1, 2, 3$.

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Theorem (Levinson, 2011)

If $\hat{f}_1, \dots, \hat{f}_n$ are supported in $\sum_{i=1}^n |u_i| < 2$, then equality holds in $(*)$ for $n = 4, 5, 6$.

In Need of Support

- RMT and NT already computed; equality for $n \leq 3$.

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1. If $\hat{f}_1$ is supported in $(-1, 1)$, then

$$\int_{|u|>1} \hat{f}_1(u)du = 0.$$

More support $\leftrightarrow$ less cancellation

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2. Combinatorial / Fourier obstructions:

$$\int_{|u|>1} \widehat{fg}(u)du = \int \int_{|u|>1} \hat{f}(v)\hat{g}(v-u)dvdu$$

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$$\int_{\mathbb{R}^2} (1-\chi(u+v)\chi(u-v))\widehat{f_1 f_2}(u)\widehat{f_3 f_4}(v)dudv$$

$$\Downarrow \text{ shift to } \mathbb{R}^4, \text{ change variables}$$

$$\int_{\mathbb{R}^4} (1-\chi(u_1+u_2+u_3+u_4)\chi(u_1+u_2-u_3-u_4))\prod_{i=1}^{4}\hat{f}_i(u_i)du_i.$$

Current efforts

Two goals:

- **Counting** pieces that appear identically in both expressions
- **Establishing** equality between remaining pieces

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- Questions?