The Distribution of Generalized Ramanujan Primes

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Joint Mathematics Meetings
January 6, 2012

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- We expect linearly increasing intervals, $(cx, x]$, to contain an increasing amount of primes.

Historical Introduction

Bertrand's Postulate (1845)

For all integers $x \geq 2$, there exists at least one prime in $(x/2, x]$.

Ramanujan Primes

Definition

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- Sondow: $R_n \sim p_{2n}$.
- Sondow: As $n \to \infty$, 50% of primes are Ramanujan.

c-Ramanujan Primes

Definition

The n-th c-Ramanujan prime $R_{c,n}$: smallest integer such that for any $x \geq R_{c,n}$ have at least n primes in $(cx, x]$ for $c \in (0, 1)$.

Preliminaries

The logarithmic integral function $\mathrm{Li}(x)$ is defined by

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The Prime Number Theorem gives us

$$\pi(x) = \mathrm{Li}(x) \; + \; O\left(\frac{x}{\log^2 x}\right),$$

i.e., there is a $C > 0$ such that for all x sufficiently large

$$-C\frac{x}{\log x} \; \leq \; \pi(x) - \mathrm{Li}(x) \; \leq \; C\frac{x}{\log x}.$$

Existence of $R_{c,n}$

Theorem (ABMRS 2011)

For all $n \in \mathbb{Z}$ and all $c \in (0, 1)$, the n-th c-Ramanujan prime $R_{c,n}$ exists.

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- The number of primes in $(cx, x]$ is $\pi(x) - \pi(cx)$.
- Using the Prime Number Theorem and Mean Value Theorem, there exists a $b_c \in [0, -\log c]$,

$$\pi(x) - \pi(cx) \ = \ \frac{(1-c)x}{\log x - b_c} + O\left(\frac{x}{\log^2 x}\right).$$

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- For sufficiently large x, $\pi(x) - \pi(cx)$ is strictly increasing and $\pi(x) - \pi(cx) \geq n$, for all integers n.

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Theorem (ABMRS 2011)

For any fixed $c \in (0, 1)$, the n-th c-Ramanujan prime is asymptotic to the $\frac{n}{1-c}$-th prime as $n \to \infty$.

Prime Numbers

2	3	5	7	11	13	17
19	23	29	31	37	41	43
47	53	59	61	67	71	73
79	83	89	97	101	103	107
109	113	127	131	137	139	149
151	157	163	167	173	179	181
191	193	197	199	211	223	227

Ramanujan Primes ($c = 1/2$)

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Distribution of generalized Ramanujan primes

c	Length of the longest run in $[10^5, 10^6]$ of			
	c-Ramanujan primes		Non-*c*-Ramanujan primes	
	Expected	Actual	Expected	Actual
0.50	14	20	16	36

Distribution of generalized c-Ramanujan primes

	Length of the longest run in $[10^5, 10^6]$ of			
	c-Ramanujan primes		Non-c-Ramanujan primes	
c	Expected	Actual	Expected	Actual
0.10	70	58	5	3
0.20	38	36	7	7
0.30	25	25	10	12
0.40	18	21	13	16
0.50	14	20	16	36
0.60	11	17	22	42
0.70	9	14	30	78
0.80	7	9	46	154
0.90	5	11	91	345

Acknowledgments

This work was supported by the NSF, Williams College, University College London.

We would like to thank our advisors Steven J. Miller and Jonathan Sondow, as well as our colleagues from the 2011 REU at Williams College.