Distribution of Missing Sums in Sumsets

Oleg Lazarev, Princeton University
Advisor: Steven Miller, Williams College
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- Goldbach's conjecture: $\{4, 6, 8, \cdots\} \subseteq P + P$.
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Key Question: What is the structure of $A + A$?

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Note: Both theorems can be more naturally stated in terms of missing sums (independent of n).

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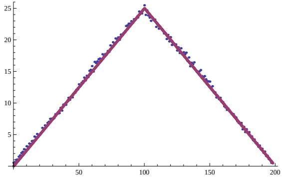

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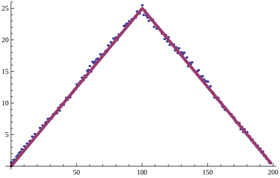

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- Key fact: if $k < n$, then $\mathrm{P}(k \notin A+A) \sim \left(\frac{3}{4}\right)^{k/2}$.

New Results

Theorem: Bounds on the distribution (Lazarev-Miller, 2011)

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Main idea: Use graph theory.

More Results

Theorem: Variance (Lazarev-Miller)

$$\mathrm{Var}|A+A| = 4 \sum_{i<j\leq n-1} P(i \text{ and } j \notin A+A) - 40 \sim 35.98.$$

Theorem: Distribution of configurations (Lazarev-Miller)

For any fixed $a_1, \cdots, a_m$, exists $\lambda_{a_1,\cdots,a_m}$ such that

$$P(k+a_1, k+a_2, \cdots, \text{ and } k+a_m \notin A+A) = \Theta(\lambda^k_{a_1,\cdots,a_m}).$$

Theorem: Consecutive missing sums (Lazarev-Miller)

$$P(k, k+1, \cdots, \text{ and } k+m \notin A+A) = \left(\frac{1}{2} + o(1)\right)^{(k+m)/2}.$$

Bound on Distribution: Upper Bound

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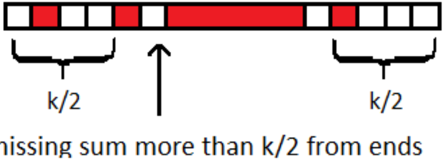

- $P(A + A \text{ has } k \text{ missing sums}) < P(k/2 \notin A + A) < \left(\frac{3}{4}\right)^{k/4} \sim 0.93^k$.

Note: Bounds on $P(k + a_1, k + a_2, \cdots, \text{ and } k + a_m \notin A + A)$ yield upper bounds on $P(A + A \text{ has } k \text{ missing sums})$.

Problem: Dependent Random Variables

Variances reduces to $\sum_{0 \leq i,j \leq 2n-2} \mathrm{P}(A : i \text{ and } j \notin A + A)$.

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- Conditions:

$i = 3$: 0 or 3 $\notin A$ and 1 or 2 $\notin A$

$j = 7$: 0 or 7 $\notin A$ and 1 or 6 $\notin A$ and 2 or 5 $\notin A$ and 3 or 4 $\notin A$.

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- Conditions:

$$i = 3 : \quad 0 \text{ or } 3 \notin A \text{ and } 1 \text{ or } 2 \notin A \qquad j = 7 : \quad 0 \text{ or } 7 \notin A \text{ and } 1 \text{ or } 6 \notin A \text{ and } 2 \text{ or } 5 \notin A \text{ and } 3 \text{ or } 4 \notin A.$$

- Since there are common integers in both lists, the events $3 \notin A + A$ and $7 \notin A + A$ are dependent.

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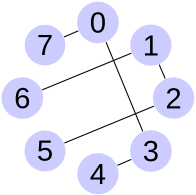

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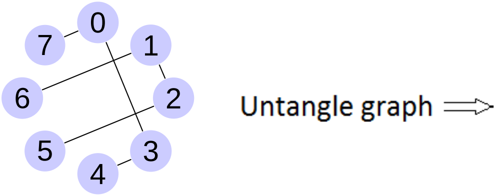

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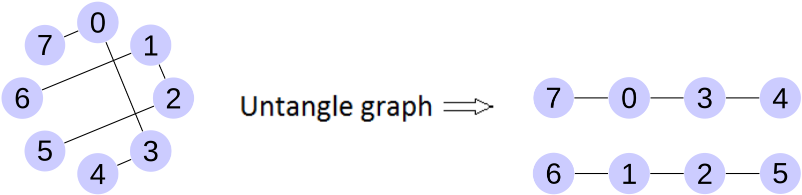

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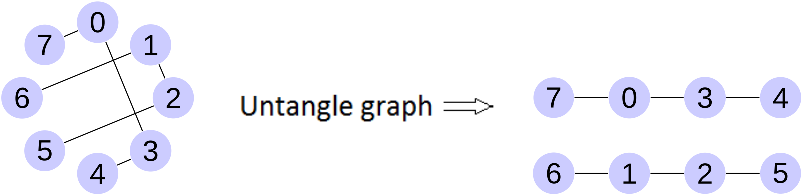

- One-to-one correspondence between conditions/edges (and integers/vertices).

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- So need to pick a vertex cover!

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Lemma (Lazarev-Miller)

$$P(i, j \notin A + A) = P(\text{pick a vertex cover for graph}).$$

Number of Vertex Covers

Condition graphs are always ‘segment’ graphs. So we just need $g(n)$, the number of vertex covers for a ‘segment’ graph with n vertices.

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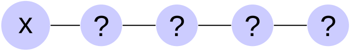

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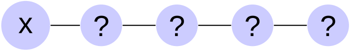

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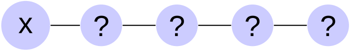

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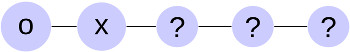

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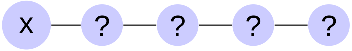

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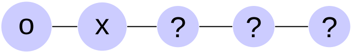

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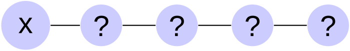

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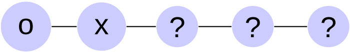

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$$\implies g(n) = F_{n+2}$$

General i, j

In particular

$$\text{P}(3 \text{ and } 7 \notin A + A) = \frac{1}{2^8} F_{4+2} F_{4+2} = \frac{1}{4}$$

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For odd $i < j < n$:

$$\begin{aligned} &P(A : i \text{ and } j \not\in A + A) \\ &= \frac{1}{2^{j+1}} F_{2\left\lceil \frac{i+1}{j-i} \right\rceil + 2}^{\frac{1}{2}\left((j-i)\left\lceil \frac{i+1}{j-i} \right\rceil - (i+1)\right)} \times F_{2\left\lceil \frac{i+1}{j-i} \right\rceil + 4}^{\frac{1}{2}\left(j+1-(j-i)\left\lceil \frac{i+1}{j-i} \right\rceil\right)} \end{aligned}$$

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In general $P(k \text{ and } k+1 \notin A + A) < C(\phi/2)^k \sim 0.81^k$, giving upper bound.

Summary

Use graph theory to study $P(a_1, \cdots, \text{ and } a_m \notin A + A)$.

Currently investigating:

- Is distribution of missing sums approximately exponential?
- Can A such that $A + A$ has k missing elements be modeled by a different random variable?
- Higher moments: third moment involves $P(i, j, k \notin A + A)$, with more complicated graphs.
- Distribution of $A - A$.

Thanks to:

- AMS / MAA
- Princeton University
- Williams College
- National Science Foundation

Thank you!