# Virus Dynamics on Star Graphs: A Generalized Model

Thealexa Becker (tbecker@smith.edu)
Smith College

Work done at 2011 SMALL REU at Williams College with Alec Greaves-Tunnell and Steven J. Miller (Williams College) and Karen Shen (Stanford University)

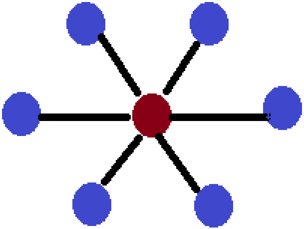
**JMM, Boston, MA, January 7, 2012**

## Motivation for the Problem

- Epidemiology, networking, and other fields have questions concerning the spread of viruses.
- Using a model for infection and cure rates, look for a steady state, or critical threshold relating two rates.
- If there is a steady state, what are the characteristics?
- What other information can we get from this steady state, provided it exists?
- Generalizations?

# The Model

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).



## The Model

A discrete-time SIS (Susceptible Infected Susceptible) model. Each node is either Susceptible (S) or Infected (I).

### Parameters

- $\beta$: probability at any time an infected node infects its neighbors.
- $\delta$: probability at any time an infected node is cured.
- $\zeta_{i,t}$: probability node $i$ not infected by neighbors at time $t$.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$.
- $\zeta_{i,t} = \prod_{j\sim i} p_{j,t-1}\,(1-\beta) + (1 - p_{j,t-1}) = \prod_{j\sim i} 1 - \beta p_{j,t-1}$, where $j \sim i$ says $i$ and $j$ are neighbors, connected by edge of the graph.
- $1 - p_{i,t} = (1 - p_{i,t-1})\,\zeta_{i,t} + \delta p_{i,t}\zeta_{i,t}$.

## A Different View of the Model

- Let's consider specific graph topology of a star graph, then we can alter the model to one of "hubs" and "spokes"
- Suppose a graph has $n+1$ nodes, the hub is numbered 0 and the spokes are numbered 1 through $n$.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = F \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

where

$$\begin{aligned} F\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-(1-x)(1-\beta y)^n - \delta x(1-\beta y)^n \\ 1-(1-y)(1-\beta x) - \delta y(1-\beta x) \end{pmatrix} \\ &= \begin{pmatrix} 1-(1-ax)(1-by)^n \\ 1-(1-ay)(1-bx) \end{pmatrix}. \end{aligned}$$

# Our Main Result

## Theorem (Main Result)

*Let $a, b \in (0, 1)$ and $F$ as above.*

- *For any initial configuration, as time evolves all the spokes converge to a common behavior.*
- *If $b \le (1 - a)/\sqrt{n}$ then the virus dies out.*
- *If $b > (1 - a)/\sqrt{n}$ then all points except $(0, 0)$ evolve to a unique, non-trivial fixed point $(x_f, y_f)$.*

## Determining Fixed Points of F

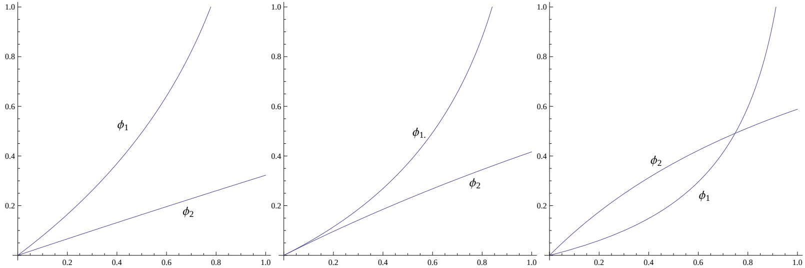



**Figure:** Partial fixed points from $\phi_1$ and $\phi_2$ when (from left to right) $b < \frac{1-a}{\sqrt{n}}$, $b = \frac{1-a}{\sqrt{n}}$, $b > \frac{1-a}{\sqrt{n}}$ ($b = 3, n = 4, a = .1, .4, .7$).

$$\phi_1(y) = \frac{1-(1-by)^n}{1-a(1-by)^n}$$

$$\phi_2(x) = \frac{bx}{1-a+abx}.$$

## Determining Fixed Points of F

### Theorem

*Consider the map $F$.*

1. *If $b < \frac{1-a}{\sqrt{n}}$ then the trivial fixed point is the unique fixed point in $[0,1]^2$.*
2. *If $b > \frac{1-a}{\sqrt{n}}$ then there exists a unique non-trivial fixed point in $[0,1]^2$.*

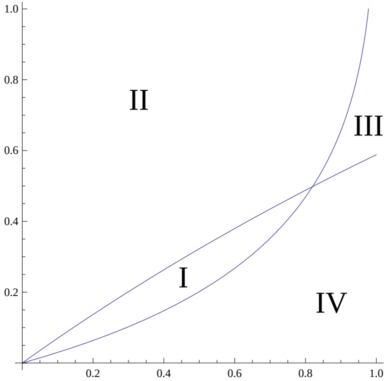
## Convergence Case $b \leq \frac{(1-a)}{\sqrt{n}}$

**Theorem**

*Assume* $b < (1-a)/\sqrt{n}$*. Then iterates of any point under* $F$ *converge to the trivial fixed point* $(0,0)$*.*

Proved this with the Mean Value Theorem and an eigenvalue analysis of the resulting matrix.

Looking at partial fixed points of $F$ divides $[0,1]^2$ into four regions.



## Convergence: Case $b > \frac{(1-a)}{\sqrt{n}}$

### Lemma

*Points in Region I strictly increase in $x$ and $y$ on iteration by $F$, and points in Region III strictly decrease in $x$ and $y$ on iteration.*

### Lemma

*Points in Region I iterate inside Region I under $F$, and points in Region III iterate inside Region III under $F$.*

### Lemma

*All non-trivial points in Regions I and III converge to the non-trivial fixed point under $F$.*

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

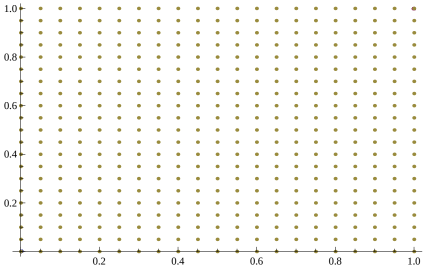



**Figure:** $t = 0$ (point in upper right for display purposes: 1/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

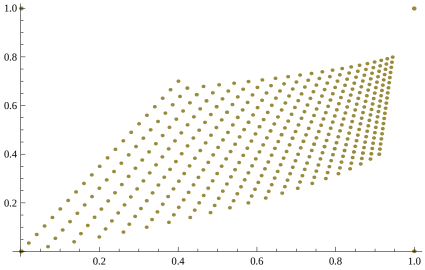



**Figure:** $t = 1$ (point in upper right for display purposes: 2/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

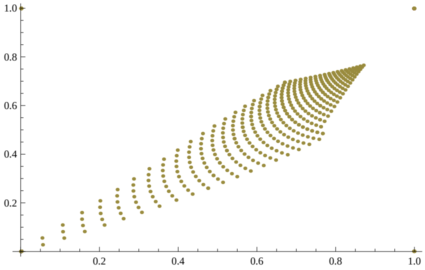



**Figure:** $t = 2$ (point in upper right for display purposes: 3/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

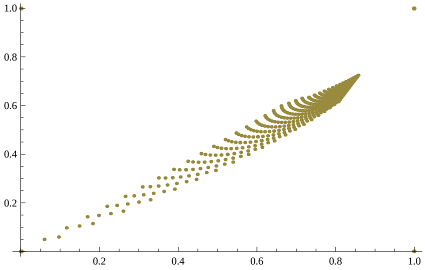



**Figure:** $t = 3$ (point in upper right for display purposes: 4/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

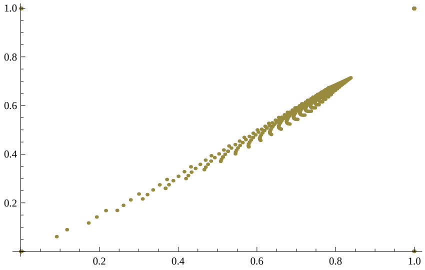



**Figure:** $t = 4$ (point in upper right for display purposes: 5/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

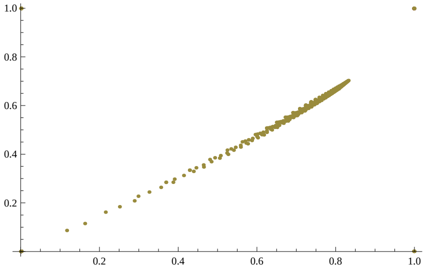



**Figure:** $t =$ 5 (point in upper right for display purposes: 6/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

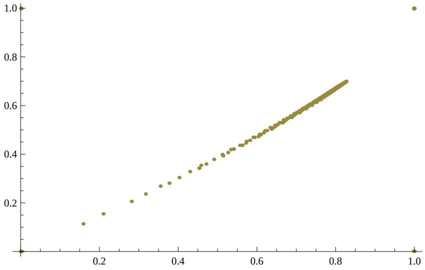



**Figure:** $t =$ 6 (point in upper right for display purposes: 7/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

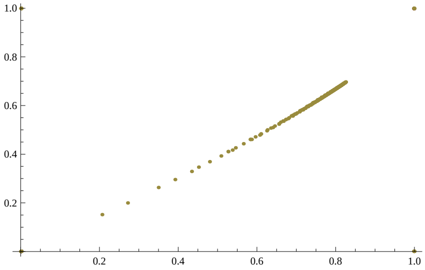



**Figure:** $t =$ 7 (point in upper right for display purposes: 8/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

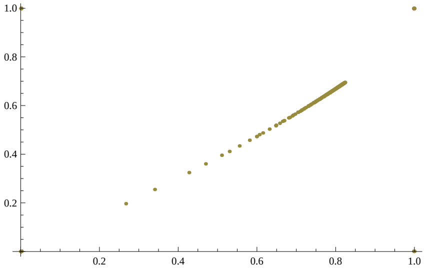



**Figure:** $t = 8$ (point in upper right for display purposes: 9/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

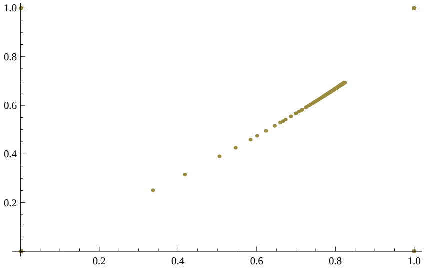



**Figure:** $t = 9$ (point in upper right for display purposes: 10/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

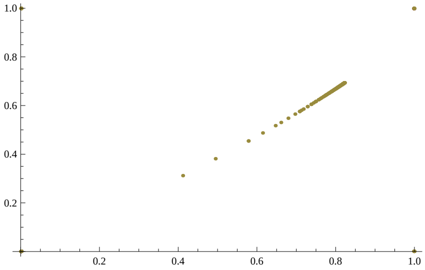



**Figure:** $t = 10$ (point in upper right for display purposes: 11/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

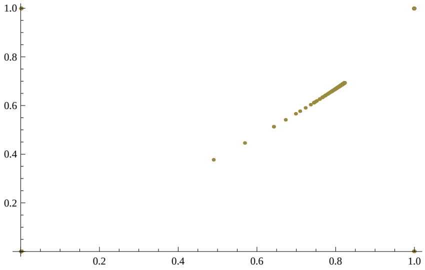



**Figure:** $t = 11$ (point in upper right for display purposes: 12/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

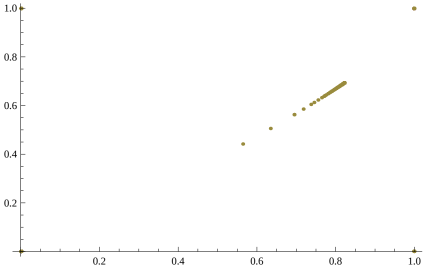



**Figure:** $t = 12$ (point in upper right for display purposes: 13/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

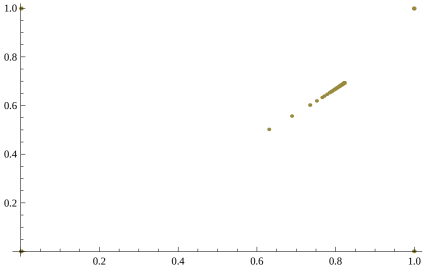



**Figure:** $t = 13$ (point in upper right for display purposes: 14/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

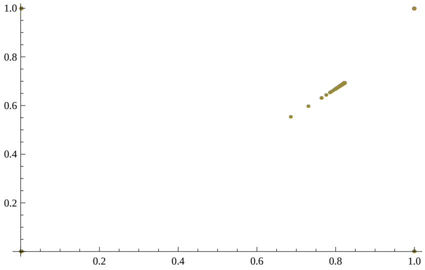



**Figure:** $t = 14$ (point in upper right for display purposes: 15/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

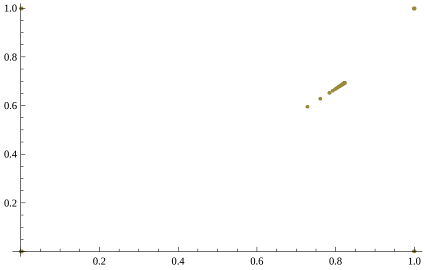



**Figure:** $t = 15$ (point in upper right for display purposes: 16/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

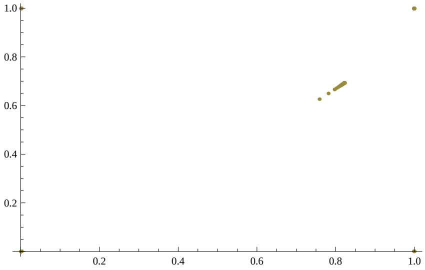



**Figure:** $t = 16$ (point in upper right for display purposes: 17/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

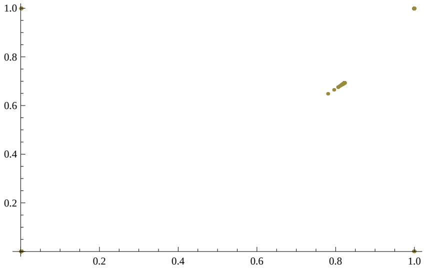



**Figure:** $t = 17$ (point in upper right for display purposes: 18/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

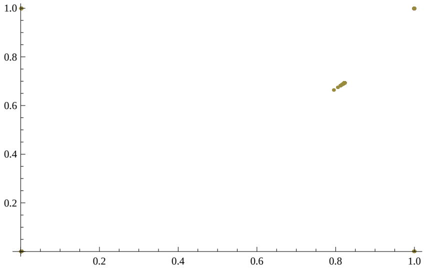



**Figure:** $t = 18$ (point in upper right for display purposes: 19/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

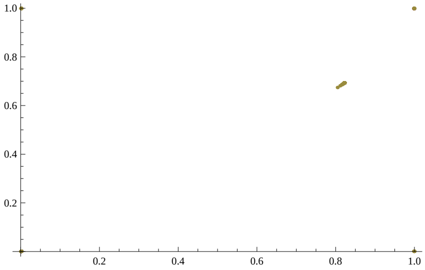



**Figure:** $t = 19$ (point in upper right for display purposes: 20/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

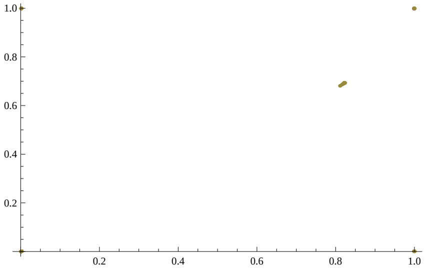



**Figure:** $t = 20$ (point in upper right for display purposes: 21/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

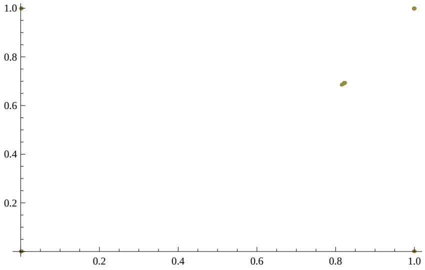



**Figure:** $t = 21$ (point in upper right for display purposes: 22/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

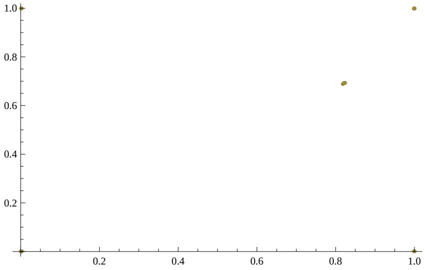



**Figure:** $t = 22$ (point in upper right for display purposes: 23/24)

## Simulation

In the limit all spokes have same behavior. Below is a plot of the dynamics for $a = .4$, $b = .7$ and $n = 2$.

Divide $(x, y)$ space into a grid, each gridpoint a different initial configuration, iterate.

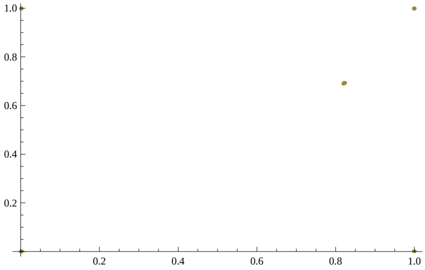



**Figure:** $t = 23$ (point in upper right for display purposes: 24/24)

**Theorem**

*Any non-trivial point in* $[0,1]^2$ *converges to the unique non-trivial fixed point under* $F$.

## Behavior Conjectures

### Corollary

*The amount of time it takes for all points to converge is the maximum of the time it takes* $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *to converge, for* $\epsilon_1, \epsilon_2 \to 0$.

### Conjecture

*Points in Region II and IV exhibit one of two behaviors, dependent on* $a, b, n$. *Either:*

1. *All points in Region II iterate outside Region II and all points in Region IV iterate outside Region IV ("flipping behavior"), or*
2. *All points in Region II iterate outside Region IV and all points in Region IV iterate outside Region II*

## Conclusions

- Generalized Star Graphs
- Conclusions

## Thanks to:

- AMS / MAA
- Smith College
- Williams College
- National Science Foundation
- Alexander Greaves-Tunnell, Aryeh Kontorovich, Steven J. Miller, Pradeep Ravikumar and Karen Shen.