Distribution of Gaps between Summands in Zeckendorf Decompositions

Olivia Beckwith

Advisor: Steven J. Miller

http://web.williams.edu/Mathematics/sjmiller/

Joint Mathematics Meetings
January 6, 2012

Introduction

A few questions

- How can we write a number as a sum of powers of 2?
- Example: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.
- The binary representation of 2012 is 11111011100.

A few questions

- How can we write a number as a sum of powers of 2?
- Example: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.
- The binary representation of 2012 is 11111011100.
- What about powers of 3? 5? 10?

A few questions

- How can we write a number as a sum of powers of 2?
- Example: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.
- The binary representation of 2012 is 11111011100.
- What about powers of 3? 5? 10?
- What other sequences can we use besides powers?

The Fibonacci Numbers

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;

The Fibonacci Numbers

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

The Fibonacci Numbers

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

Generating function: $g(x) := \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$.

The Fibonacci Numbers

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

Generating function: $g(x) := \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$.

Binet's Formula:

$$F_n = \frac{\phi^{n+1} - (1-\phi)^{n+1}}{\sqrt{5}},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Previous Results

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:

$$2012 = 1597 + 377 + 34 + 3 = F_{16} + F_{13} + F_8 + F_3 + F_1.$$

Previous Results

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:
$2012 = 1597 + 377 + 34 + 3 = F_{16} + F_{13} + F_8 + F_3 + F_1$.
Fibonacci representation of 2012 is 1001000010000101.

Previous Results

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:
$2012 = 1597 + 377 + 34 + 3 = F_{16} + F_{13} + F_8 + F_3 + F_1$.
Fibonacci representation of 2012 is 1001000010000101.

Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\phi^2+1} \approx .276n$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ converges to a Gaussian (normal).

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \cdots, i_2 - i_1$.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \cdots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 10 and 7.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \cdots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 10 and 7.

Define $P_n(k)$ to be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length k.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \cdots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 10 and 7.

Define $P_n(k)$ to be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length k.

What is $P(k) := \lim_{n\to\infty} P_n(k)$?

Main Results

Zeckendorf Gap Distribution (Beckwith-Miller)

The percent of gaps of length k in Zeckendorf decompositions is given by $P(k) = (\phi - 1)\phi^{1-k}$.

Main Results

Zeckendorf Gap Distribution (Beckwith-Miller)

The percent of gaps of length k in Zeckendorf decompositions is given by $P(k) = (\phi - 1)\phi^{1-k}$.

Base B Distribution (Beckwith-Miller)

The percent of gaps of length k in base B decompositions is $P(k) = c_B B^{-k}$, where $c_B = \frac{2(B-1)^2}{B^2-2}$ for $k \geq 1$, and $P(0) = \frac{B(B-2)}{B^2-2}$.

Proof of Fibonacci Result

What is the probability that a gap in a decomposition in $[F_n, F_{n+1})$ has length k?

Proof of Fibonacci Result

What is the probability that a gap in a decomposition in $[F_n, F_{n+1})$ has length k?

By Lekkerkerker, total number of gaps approximately $F_{n-1}\frac{n}{\phi^2+1}$.

Proof of Fibonacci Result

What is the probability that a gap in a decomposition in $[F_n, F_{n+1})$ has length k?

By Lekkerkerker, total number of gaps approximately $F_{n-1}\frac{n}{\phi^2+1}$.

Let $x_{i,j} = \left|\{m \in [F_n, F_{n+1})\text{: decomposition of } m \text{ includes } F_i, F_j\text{, but not } F_q \text{ for } i < q < j\}\right|$

$$P_n(k) = \frac{\sum_{i=1,n-k} x_{i,i+k}}{F_{n-1}\frac{1}{\phi^2+1}}$$

Calculating $x_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

Calculating $x_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

$1 \leq i \leq n - k - 2$:

Calculating $x_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

$1 \leq i \leq n - k - 2$:

For the indices less than i: F_{i-1} choices.

For the indices greater than $i + k$: $F_{n-k-2-i}$ choices.

Calculating $x_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

$1 \leq i \leq n-k-2$:

For the indices less than i: F_{i-1} choices.

For the indices greater than $i+k$: $F_{n-k-2-i}$ choices.

So total number of choices is $F_{n-k-2-i}F_{i-1}$.

Calculating $x_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

$1 \leq i \leq n-k-2$:

For the indices less than i: F_{i-1} choices.

For the indices greater than $i+k$: $F_{n-k-2-i}$ choices.

So total number of choices is $F_{n-k-2-i}F_{i-1}$.

$i = n-k-1$: 0.

$i = n-k$: F_{n-k-1}.

Proof

$$\sum_{1\le i\le n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$
$$= F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$$

Proof

$$\sum_{1 \le i \le n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1} F_{n-k-i-2}$$
$$= F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$$

$\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $g(x)^2$.

Proof

$$\sum_{1\le i\le n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$
$$= F_{n-k-1} + \sum_{i=0}^{n-k-3} F_iF_{n-k-i-3}$$

$\sum_{i=0}^{n-k-3} F_iF_{n-k-i-3}$ is the x^{n-k-3} coefficient of $g(x)^2$.

$g(x)^2 = (\frac{x}{1-x-x^2})^2$

$= \frac{A}{(1-\phi x)^2} + \frac{B}{(1-x(1-\phi))^2} + \frac{C}{1-\phi x} + \frac{D}{1-(1-\phi)x}$.

Proof

$$\sum_{1\leq i\leq n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$
$$= F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$$

$\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $g(x)^2$.
$g(x)^2 = (\frac{x}{1-x-x^2})^2$
$= \frac{A}{(1-\phi x)^2} + \frac{B}{(1-x(1-\phi))^2} + \frac{C}{1-\phi x} + \frac{D}{1-(1-\phi)x}$.
By the geometric series formulas, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, the x^m coefficient is
$A(m+1)\phi^m + B(m+1)(1-\phi)^m + C\phi^m + D(1-\phi)^m$.

Consider the ratio:

$$\frac{P(k+1)}{P(k)} = \lim_{n\to\infty} \frac{P_n(k+1)}{P_n(k)}$$

$$= \lim_{n\to\infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}$$

$$= \lim_{n\to\infty} \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}$$

$$= \frac{1}{\phi}.$$

Consider the ratio:

$$\frac{P(k+1)}{P(k)} = \lim_{n\to\infty} \frac{P_n(k+1)}{P_n(k)}$$

$$= \lim_{n\to\infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}$$

$$= \lim_{n\to\infty} \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}$$

$$= \frac{1}{\phi}.$$

So, the distribution is a geometric series!

Consider the ratio:

$$\begin{aligned}
\frac{P(k+1)}{P(k)} &= \lim_{n\to\infty} \frac{P_n(k+1)}{P_n(k)} \\
&= \lim_{n\to\infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}} \\
&= \lim_{n\to\infty} \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}} \\
&= \frac{1}{\phi}.
\end{aligned}$$

So, the distribution is a geometric series!
Therefore there is a constant C so that $P(k) = C/\phi^k$.

Consider the ratio:

$$\frac{P(k+1)}{P(k)} = \lim_{n\to\infty} \frac{P_n(k+1)}{P_n(k)}$$
$$= \lim_{n\to\infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}}$$
$$= \lim_{n\to\infty} \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}}$$
$$= \frac{1}{\phi}.$$

So, the distribution is a geometric series!
Therefore there is a constant C so that $P(k) = C/\phi^k$.
Thus $1 = C\sum_{k\geq 2}\phi^{-k} = C/\phi^2(1-1/\phi)$, so $C = \phi(\phi-1)$, and $P(k) = (\phi-1)/\phi^{k-1}$.

Acknowledgements

Thanks to:

- Williams College
- National Science Foundation
- Professor Steven Miller