Theoretical Physics Seminar

From the Manhatten Project to Number Theory: How Nuclear Physics helps us understand primes

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Acknowledgements

Random Matrices (with Peter Sarnak, Eduardo Dueñez)

- Rebecca Lehman
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Random Graphs (with Peter Sarnak, Yakov Sinai)

- Dustin Steinhauer
- Randy Qian
- Felice Kuan
- Leo Goldmakher
Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \ldots$.

Question: what rules govern the spacings between the $t_i$?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w Zeros of Functions.
Goals of the Talk

• Determine correct scale to study spacings.

• See similar behavior in different systems.

• Discuss tools / techniques needed to prove the results.
PART I

NORMALIZED SPACINGS
Normalized Spacing

Example: Fractional Parts

For $\alpha \not\in \mathbb{Q}$, set $x_n = n\alpha \mod 1$.

Order $x_1, \ldots, x_N$: $0 \leq y_1 \leq \cdots \leq y_N \leq 1$.

Expect spacings between adjacent $y$’s of size $\frac{1}{N}$.

Should study $\frac{y_{n+1} - y_n}{1/N}$.
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium (200+ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

\[ H \psi_n = E_n \psi_n \]

\( H \) : matrix, entries depend on system
\( E_n \) : energy levels
\( \psi_n \) : energy eigenfunctions
Origins (continued)

Statistical Mechanics: for each configuration, calculate quantity (say pressure).

Average over all configurations – most configurations close to system average.

Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices.

Look at: Real Symmetric \((A = A^T)\), Complex Hermitian \((\overline{A}^T = A)\).
Random Matrix Ensembles

Real Symmetric Matrices:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN}
\end{pmatrix} = A^T, \quad a_{ij} = a_{ji}
\]

Fix \( p \), define

\[
\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).
\]

This means

\[
\text{Prob} \left( A : a_{ij} \in [\alpha_{ij}, \beta_{ij}] \right) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) \, dx_{ij}.
\]

Want to understand eigenvalues of \( A \).
Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at $x_0$.

To each $A$, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$

$$\int_{a}^{b} \mu_{A,N}(x) dx = \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}$$

$k$th moment

$$= \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{k+1}} = \frac{\text{Trace}(A^k)}{2^k N^{k+1}}.$$
Wigner’s Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$.

**Semi-Circle Law:** Assume $p$ has mean 0, variance 1, other moments finite. Then for almost all $A$, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
SKETCH OF PROOF: Correct Scale

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.
\]

By the Central Limit Theorem:

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2
\]

\[
\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2
\]

Gives \(N \text{Ave}(\lambda_i(A)^2) \sim N^2\) or \(\text{Ave}(\lambda_i(A)) \sim \sqrt{N}\).
SKETCH OF PROOF: Method of Moments

\[ \text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2. \]

Substituting into expansion gives

\[ \frac{1}{2^2N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN} \]

Integration factors as

\[ \int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \cdot \prod_{(k,l) \neq (i,j), k<l} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} = 1. \]

Have \( N^2 \) summands, answer is \( \frac{1}{4} = 2^{\text{nd}} \) moment of Semi-Circle.
Random Matrix Theory: Semi-Circle Law

500 Matrices: Gaussian $400 \times 400$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
The eigenvalues of the Cauchy distribution are NOT semicircular.

Cauchy Distribution: \( p(x) = \frac{1}{\pi(1+x^2)} \)
GOE Conjecture

**GOE Conjecture:** As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of $p$.

Only known if $p$ is a Gaussian.

$$\text{GOE}(x) \approx Ax e^{-Bx^2}.$$
Uniform Distribution: \( p(x) = \frac{1}{2} \) for \( |x| \leq 1 \)

The local spacings of the central 3/5 of the eigenvalues of 5000 300x300 uniform matrices, normalized in batches of 20.

5000: 300 \times 300 uniform on \([-1, 1]\)
Cauchy Distribution: \[ p(x) = \frac{1}{\pi(1+x^2)} \]
Fat Thin Families

Need a family *F A T* enough to do averaging.

Need a family *THIN* enough so that everything isn’t averaged out.

Real Symmetric Matrices have \( \frac{N(N+1)}{2} \) independent entries.
Random Graphs

Degree of a vertex = number of edges leaving the vertex.

Adjacency matrix: \( a_{ij} = \) number edges b/w Vertex \( i \) and Vertex \( j \).

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 2 \\
1 & 0 & 2 & 0
\end{pmatrix}
\]

These are Real Symmetric Matrices.
McKay’s Law (Kesten Measure)

Density of Eigenvalues for $d$-regular graphs

$$f(x) = \begin{cases} 
\frac{d}{2\pi(d^2-x^2)} \sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\
0 & \text{otherwise.}
\end{cases}$$

$d = 3$. 
McKay’s Law (Kesten Measure)

\[ d = 6. \]

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.
3-Regular, 2000 Vertices and GOE
PART III

NUMBER THEORY
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^s} \right)^{-1}, \quad \text{Re}(s) > 1. \]

Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)

\[ \prod_{p} \left( 1 - \frac{1}{p^s} \right)^{-1} = \left[ 1 + \frac{1}{2^s} + \left( \frac{1}{2^s} \right)^2 + \cdots \right] \left[ 1 + \frac{1}{3^s} + \left( \frac{1}{3^s} \right)^2 + \cdots \right] \cdots \]

\[ = \sum_{n} \frac{1}{n^s}. \]
Riemann Zeta Function (cont):

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime, } p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1+} \zeta(s) = \infty, \pi(x) \to \infty. \)

- \( \zeta(2) = \frac{\pi^2}{6}, \pi(x) \to \infty. \)
Riemann Zeta Function (cont):

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Functional Equation:

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1 - s). \]

Riemann Hypothesis (RH):

All non-trivial zeros have \( \text{Re}(s) = \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).

Observation:

Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices \( (A^T = A) \).
Zeros of $\zeta(s)$ vs GUE

70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20}$th zero (from Odlyzko)
Explicit Formula: (Contour Integration)

\[
\frac{-\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = \frac{d}{ds} \sum_p \log \left(1 - p^{-s}\right)
\]

\[
= \sum_p \log p \cdot \frac{p^{-s}}{1 - p^{-s}}
\]

\[
= \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\]

Contour Integration:

\[
\int -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
\]

Knowledge of zeros gives info on coefficients.
PART IV

ELLIPTIC CURVES
Elliptic Curves:

\[ E_t : y^2 = x^3 + A(T)x + B(T), \quad A(T), B(T) \in \mathbb{Z}(T). \]

\[ a_t(p) = -\sum_{x \mod p} \left( \frac{x^3 + A(t)x + B(t)}{p} \right) = a_{t+mp}(p) \]

\[ L(E, s) = \sum_{n=1}^{\infty} \frac{a_{E}(n)}{n^s} = \prod_p L_p(E, s). \]

By GRH: All zeros on the critical line.

Rational solutions: \( E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T. \)

Birch and Swinnerton-Dyer Conjecture:
Geometric rank \( r = \) analytic rank (order of vanishing at central point).
Density Conjecture: Distribution of low zeros of $L$-functions agree with the distribution of eigenvalues near 1 of a classical compact group.
Tools to Study Low Zeros

• explicit formula relating zeros and Fourier coeffs;
  ◦ Analogue of Eigenvalue Trace Lemma

• averaging formulas for the family;
  ◦ Analogue of integration formulas for Trace($A^k$).
Explicit Formula

\[
\frac{1}{|F|} \sum_{E \in \mathcal{F}} \sum_j \phi \left( \frac{\log N_E \gamma_E^{(j)}}{2\pi} \right) = \frac{1}{|F|} \sum_{E \in \mathcal{F}} \widehat{\phi}(0) + \phi_i(0) \\
- \frac{2}{|F|} \sum_{E \in \mathcal{F}} \sum_p \log p \frac{1}{\log N_E p} \phi \left( \frac{\log p}{\log N_E} \right) a_E(p) \\
- \frac{2}{|F|} \sum_{E \in \mathcal{F}} \sum_p \log p \frac{1}{\log N_E p^2} \phi \left( 2 \frac{\log p}{\log N_E} \right) a_E^2(p) \\
+ O \left( \frac{\log \log N_E}{\log N_E} \right)
\]

Want to move \( \frac{1}{|F|} \sum_{E \in \mathcal{F}} \), Leads us to study

\[ A_{r, \mathcal{F}}(p) = \sum_{t \mod p} a_t^r(p), \quad r = 1 \text{ or } 2. \]
One-Level Result

For small support, one-param family of rank $r$ over $\mathbb{Q}(T)$:

$$\lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi \left( \frac{\log N_E}{2\pi} \gamma_E(j) \right) = \int \phi(x) W_{\mathcal{G}}(x) dx + r \phi(0),$$

where

$$\mathcal{G} = \begin{cases} 
O & \text{if half odd} \\
\text{SO(even)} & \text{if all even} \\
\text{SO(odd)} & \text{if all odd}
\end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Forced zeros seem independent.
Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}$.

Natural sub-families

• Curves of rank $r$.
• Curves of rank $r + 2$.

**Question:** Does the sub-family of rank $r + 2$ curves in a rank $r$ family behave like the sub-family of rank $r + 2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?
Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i \theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j.$$ 

**Independent Model:**

$$A_{2N,2r} = \left\{ \begin{pmatrix} I_{2r} \times 2r \\ g \end{pmatrix} : g \in SO(2N - 2r) \right\}. $$

**Interaction Model:**

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\epsilon_{2r}(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j (1 - \cos \theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \leq j, k \leq N - r$. 
Comparing the RMT Models

Small support, one-parameter families agree with $\rho_{r,\text{Indep}}$ and not $\rho_{r,\text{Inter}}$.

Curve $E$, conductor $N_E$, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log N_E}$.

$r$ zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log N_E}$, can detect in

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi \left( \frac{\gamma_E^{(j)} \log N_E}{2\pi} \right).$$

Corrections of size

$$\phi(x_0 + c_r) - \phi(x_0) \approx \phi'(x(x_0, c_r)) \cdot c_r.$$
Testing Random Matrix Theory Predictions

First (Normalized) Zero above Central Point: Do extra zeros at the central point affect the distribution of zeros near the central point?
RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

Figure 1a: First normalized eigenangle above 0: 23,040 SO(4) matrices Mean $= .357$, Standard Deviation about the Mean $= .302$, Median $= .357$

Figure 1b: First normalized eigenangle above 0: 23,040 SO(6) matrices Mean $= .325$, Standard Deviation about the Mean $= .284$, Median $= .325$

Figure 1c: First normalized eigenangle above 0: $N \to \infty$ scaling limit of SO($2N$): Mean $= .321$. 
Figure 2a: 750 rank 0 curves from $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. 
$log(\text{cond}) \in [3.2, 12.6]$, median = 1.00, mean = 1.04, $\sigma_{\mu} = .32$

Figure 2b: 750 rank 0 curves from $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. 
$log(\text{cond}) \in [12.6, 14.9]$, median = .85, mean = .88, $\sigma_{\mu} = .27$
Rank 2 Curves: 1st Norm. Zero above the Central Point

Figure 3a: 665 rank 2 curves from $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$.  
$log(\text{cond}) \in [10, 10.3125]$, median = 2.29, mean = 2.30

Figure 3b: 665 rank 2 curves from $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$.  
$log(\text{cond}) \in [16, 16.5]$, median = 1.81, mean = 1.82
**Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0**

![Histogram](attachment:figure4a.png)

**Figure 4a:** 209 rank 0 curves from 14 rank 0 families, log(cond) ∈ [3.26, 9.98], median = 1.35, mean = 1.36

![Histogram](attachment:figure4b.png)

**Figure 4b:** 996 rank 0 curves from 14 rank 0 families, log(cond) ∈ [15.00, 16.00], median = .81, mean = .86.
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

<table>
<thead>
<tr>
<th>Family</th>
<th>Median $\tilde{\mu}$</th>
<th>Mean $\mu$</th>
<th>StDev $\sigma_\mu$</th>
<th>log(conductor)</th>
<th>Number</th>
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</thead>
<tbody>
<tr>
<td>1: [0,1,1,1,T]</td>
<td>1.28</td>
<td>1.33</td>
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<td>1.40</td>
<td>0.29</td>
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<td>3: [1,0,0,2,T]</td>
<td>1.40</td>
<td>1.41</td>
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<td>9: [1,0,-1,1,T]</td>
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<td>10: [1,0,-2,1,T]</td>
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<td>1.34</td>
<td>0.42</td>
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<td>11: [1,1,-2,1,T]</td>
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<td>1.19</td>
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<td>12: [1,1,-3,1,T]</td>
<td>1.32</td>
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<td>13: [1,-2,0,T,0]</td>
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<td>14: [-1,1,-3,1,T]</td>
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<td>1.45</td>
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<td>All Curves</td>
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Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

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</tr>
<tr>
<td>12: [1,1,-3,1,T]</td>
<td>0.98</td>
<td>0.99</td>
<td>0.36</td>
<td>[15.01, 16.00]</td>
<td>66</td>
</tr>
<tr>
<td>13: [1,-2,0,T,0]</td>
<td>0.72</td>
<td>0.76</td>
<td>0.27</td>
<td>[15.00, 16.00]</td>
<td>120</td>
</tr>
<tr>
<td>14: [-1,1,-3,1,T]</td>
<td>0.90</td>
<td>0.91</td>
<td>0.24</td>
<td>[15.00, 15.99]</td>
<td>48</td>
</tr>
<tr>
<td>All Curves</td>
<td>0.81</td>
<td>0.86</td>
<td>0.29</td>
<td>[15.00,16.00]</td>
<td>996</td>
</tr>
<tr>
<td>Distinct Curves</td>
<td>0.81</td>
<td>0.86</td>
<td>0.28</td>
<td>[15.00,16.00]</td>
<td>863</td>
</tr>
</tbody>
</table>
Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$
(Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

Figure 5a: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\tilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Figure 5b: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\tilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$
Repulsion or Attraction?

Conductors in \([15, 16]\); first set is rank 0 curves from 14 one-parameter families of rank 0 over \(\mathbb{Q}\); second set rank 2 curves from 21 one-parameter families of rank 0 over \(\mathbb{Q}\). The \(t\)-statistics exceed 6.

<table>
<thead>
<tr>
<th>Family</th>
<th>2nd vs 1st Zero</th>
<th>3rd vs 2nd Zero</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank 0 Curves</td>
<td>2.16</td>
<td>3.41</td>
<td>863</td>
</tr>
<tr>
<td>Rank 2 Curves</td>
<td>1.93</td>
<td>3.27</td>
<td>701</td>
</tr>
</tbody>
</table>

The additional repulsion from extra zeros at the central point cannot be entirely explained by only collapsing the first zero to the central point while leaving the other zeros alone.
Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;

- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

<table>
<thead>
<tr>
<th>Family</th>
<th>Median</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank 2 Curves from Rank 0 Families</td>
<td>1.926</td>
<td>1.936</td>
<td>0.388</td>
<td>701</td>
</tr>
<tr>
<td>Rank 2 Curves from Rank 2 Families</td>
<td>1.642</td>
<td>1.610</td>
<td>0.247</td>
<td>64</td>
</tr>
</tbody>
</table>

- $t$-statistic is 6.60, indicating the means differ.

- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on how we choose the curves.
Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of $j^{\text{th}}$ normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>863 Rank 0 Curves</th>
<th>701 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median $z_2 - z_1$</td>
<td>1.28</td>
<td>1.30</td>
<td></td>
</tr>
<tr>
<td>Mean $z_2 - z_1$</td>
<td>1.30</td>
<td>1.34</td>
<td>-1.60</td>
</tr>
<tr>
<td>StDev $z_2 - z_1$</td>
<td>0.49</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_2$</td>
<td>1.22</td>
<td>1.19</td>
<td></td>
</tr>
<tr>
<td>Mean $z_3 - z_2$</td>
<td>1.24</td>
<td>1.22</td>
<td>0.80</td>
</tr>
<tr>
<td>StDev $z_3 - z_2$</td>
<td>0.52</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_1$</td>
<td>2.54</td>
<td>2.56</td>
<td></td>
</tr>
<tr>
<td>Mean $z_3 - z_1$</td>
<td>2.55</td>
<td>2.56</td>
<td>-0.38</td>
</tr>
<tr>
<td>StDev $z_3 - z_1$</td>
<td>0.52</td>
<td>0.52</td>
<td></td>
</tr>
</tbody>
</table>
Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- While the normalized zeros are repelled from the central point (and by different amounts for the two sets), the *differences* between the normalized zeros are statistically independent of this repulsion ($t$-statistics $< 2$).

- While for a given range of log-conductors the average second normalized zero of a rank 0 curve is close to the average first normalized zero of a rank 2 curve, they are not the same and the additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.
Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>701 Rank 2 Curves</th>
<th>64 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median $z_2 - z_1$</td>
<td>1.30</td>
<td>1.26</td>
<td>0.69</td>
</tr>
<tr>
<td>Mean $z_2 - z_1$</td>
<td>1.34</td>
<td>1.36</td>
<td></td>
</tr>
<tr>
<td>StDev $z_2 - z_1$</td>
<td>0.51</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_2$</td>
<td>1.19</td>
<td>1.22</td>
<td>1.39</td>
</tr>
<tr>
<td>Mean $z_3 - z_2$</td>
<td>1.22</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>StDev $z_3 - z_2$</td>
<td>0.47</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_1$</td>
<td>2.56</td>
<td>2.66</td>
<td>1.93</td>
</tr>
<tr>
<td>Mean $z_3 - z_1$</td>
<td>2.56</td>
<td>2.65</td>
<td></td>
</tr>
<tr>
<td>StDev $z_3 - z_1$</td>
<td>0.52</td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>
PART V

CONCLUSIONS
Correspondences

Similarities b/w Nuclei and Primes:

Energy Levels $\longleftrightarrow$ Zeros of $L$-Functions

Neutron Energy $\longleftrightarrow$ Summation Lemmas (Test Fn Support)

Different Elements: U, Pu, ... $\longleftrightarrow$ Different $L$-Functions or Families
Summary

• Similar behavior in different systems.

• Find correct scale.

• Average over similar elements.

• Need a Trace Lemma.

• Thin subsets can exhibit very different behavior.
# Open Problems

## Real Symmetric Band Matrices

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 & 0 \\
a_{1,2} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 & 0 \\
0 & a_{2,3} & a_{3,3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{N-2,N-2} & a_{N-2,N} & 0 \\
0 & 0 & 0 & \cdots & a_{N-2,N} & a_{N-1,N-1} & a_{N-1,N} \\
0 & 0 & 0 & \cdots & 0 & a_{N-1,N} & a_{N,N}
\end{pmatrix}
\]

## Real Symmetric Toeplitz Matrices

\[
\begin{pmatrix}
b_0 & b_1 & b_2 & \cdots & b_{N-1} \\
b_1 & b_0 & b_1 & \cdots & b_{N-2} \\
b_2 & b_1 & b_0 & \cdots & b_{N-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0
\end{pmatrix}
\]

## Rates of Convergence