Evidence for a Spectral Interpretation of the Zeros of L-functions from One-Parameter Families of Elliptic Curves.

Steven Miller

Princeton University

Boston University: February 10^{th}, 2003

http://www.princeton.edu/∼sjmiller/math/talks/talks.html

Fundamental Problem: Spacing Between Events

General Formulation: Studying some system, observe values at t_1, t_2, t_3, etc. Question: what rules govern the spacings between events?

Often need to normalize by average spacing.

Example 1: Spacings Between Primes / Prime Pairs.

Example 2: Spacings Between Energy Levels of Nuclei.

Example 3: Spacings Between Eigenvalues of Matrices.

Example 4: Spacings Between Zeros of L-Functions.

Elliptic Curves

Consider $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $a_i \in \mathbb{Q}$ and its L-function

$$L(s, E) = \prod_{p|\Delta} \left(1 - a_p p^{-s}\right)^{-1} \prod_{p \dagger \Delta} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$

By GRH: All non-trivial zeros on the critical line, can talk about spacings between zeros.

Rational solutions form a group: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$, T is the torsion points, r is the geometric rank.

Birch and Swinnerton-Dyer Conjecture: Geometric rank equals the analytic rank, the order of vanishing of $L(s, E)$ at $s = \frac{1}{2}$.

One-parameter families: $a_i = a_i(t) \in \mathbb{Z}[t]$.

Random Matrix Theory

Consider the group of $N \times N$ matrices from one of the classical compact groups: unitary, symplectic, orthogonal.

One assigns probability measures to matrices from various groups. By explicitly calculating properties associated to an individual matrix and integrating over the group, one can often use the group average to make good predictions about the expected behavior of statistics from a generic, randomly chosen element.

More generally, can consider other spaces: GUE / GOE: Hermitian / Symmetric matrices with Gaussian probabilities for entries.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ be an increasing sequence of numbers, $B \subset \mathbf{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

Results:

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak): a_ps
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. insensitive to any finite set of zeros

Measures of Spacings: n-Level Density and Families

Let $f(x) = \prod_i f_i(x_i)$, f_i even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\left(L_E\gamma_E^{(j_1)}\right)\cdots f_n\left(L_E\gamma_E^{(j_n)}\right)$$

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

How should we normalize the zeros of the curves in our family?

1. Local Data (hard): using some natural measure from the curve

2. Global Data (easy): using an average from the family

Hope: for f a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} D_{n,E}(f) \;=\; \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i f_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int f(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$= \int\cdots\int \widehat{f}(u)\widehat{W_{n,\mathcal{G}(\mathcal{F})}}(u)du.$$

Much of the work is handling the dependence on the conductors.

1-Level Densities

Katz and Sarnak calculate the n-level densities for the classical compact groups. Unlike the correlations, the densities are different for different groups.

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,O^+}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,O}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,O^-}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and $\eta(u)$ is $1,\ \frac{1}{2}$, and 0 for $|u|$ less than 1, 1, and greater than 1.

2-Level Densities

We give the effect of the Fourier Transform of the densities on test functions supported in $\sigma_1 + \sigma_2 < 1$, where σ_i is the support of f_i.

Let $c(\mathcal{G}) = 0$, $\frac{1}{2}$ or 1 for $\mathcal{G} = SO(\text{even})$, O, and $SO(\text{odd})$. For $\mathcal{G}$ one of these three groups we have

$$\begin{aligned}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 &= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] \\ &\quad + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - 2\widehat{f_1 f_2}(0) \\ &\quad - f_1(0)f_2(0) \\ &\quad + c(\mathcal{G})f_1(0)f_2(0).\end{aligned}$$

For $\mathcal{G} = U$ we have

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,U}}(u)du_1du_2 = \widehat{f_1}(0)\widehat{f_2}(0) + \int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - \widehat{f_1 f_2}(0),$$

and for $\mathcal{G} = Sp$, we have

$$\begin{aligned}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 &= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] \\ &\quad + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - 2\widehat{f_1 f_2}(0) \\ &\quad - f_1(0)f_2(0) \\ &\quad - f_1(0)\widehat{f_2}(0) - \widehat{f_1}(0)f_2(0) + 2f_1(0)f_2(0).\end{aligned}$$

These densities are all distinguishable for functions with arbitrarily small support.

For the orthogonal groups, the densities (in this range) depend only on the distribution of the signs of the functional eqs.

2-Level Density (cont)

For small support, the difference between the different orthogonal densities is due to combinatorics involving the sign of the curve.

For elliptic curves, we need to subtract off $j_1 = \pm j_2$ terms. Let $\rho = 1 + i\gamma_E^{(j)}$ be a zero. For a curve with even functional equation, we may label the zeros by

$$\cdots \le \gamma_E^{(-2)} \le \gamma_E^{(-1)} \le 0 \le \gamma_E^{(1)} \le \gamma_E^{(2)} \le \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)},$$

while for a curve with odd functional equation we label the zeros by

$$\cdots \le \gamma_E^{(-1)} \le 0 \le \gamma_E^{(0)} = 0 \le \gamma_E^{(1)} \le \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)}.$$

For $j_1 \neq 0$, there are two choices for j_2. If the curve is even, we are done and subtract off $2D_{1,E}(f_1 f_2)$. If the curve is odd, there is only one choice for $j_1 = 0$, so we must add back $j_1 = j_2 = 0$, or $f_1(0)f_2(0)$.

Explicit Formula

Relates sums of test functions over zeros to sums over primes of $a_E(p)$ and $a_E^2(p)$.

$$\begin{aligned}\sum_{\gamma_E^{(j)}} G\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) &= \widehat{G}(0) + G(0) \\ &\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log N_E}\right)a_E(p) \\ &\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log N_E}\right)a_E^2(p) \\ &\quad + O\left(\frac{\log\log N_E}{\log N_E}\right).\end{aligned}$$

Modified Explicit Formula:

$$\begin{aligned}\sum_{\gamma_E^{(j)}} G\left(\frac{\log X}{2\pi}\gamma_E^{(j)}\right) &= \frac{\log N_E}{\log X}\widehat{G}(0) + G(0) \\ &\quad - 2\sum_p \frac{\log p}{\log X}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log X}\right)a_E(p) \\ &\quad - 2\sum_p \frac{\log p}{\log X}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log X}\right)a_E^2(p) \\ &\quad + O\left(\frac{\log\log X}{\log X}\right).\end{aligned}$$

Some Previous Results

1. Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
2. Symplectic: Rubinstein: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.

Main Tools:

1. Averaging Formulas (Petersson formula in ILS, Orthogonality of characters in Rubinstein).
2. Constancy of conductors.

Elliptic Curve Conductors:

$$C(t) = \prod_{p|\Delta(t)} p^{f_p(t)}$$

1-Level Expansion

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_j f\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right)$$

$$\begin{aligned} = & \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{f}(0) + f_i(0) \\ & - \frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{f}\left(\frac{\log p}{\log N_E}\right)a_E(p) \\ & - \frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{f}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p) \\ & + O\left(\frac{\log\log N_E}{\log N_E}\right)\end{aligned}$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$

Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t(p)} a_t^r(p), \quad r = 1 \text{ or } 2.$$

2-Level Expansion

Need to evaluate terms like

$$\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \prod_{i=1}^{2} \frac{1}{p_i^{r_i}} g_i \left(\frac{\log p_i}{\log N_E} \right) a_E^{r_i}(p_i).$$

Analogue of Petersson / Orthogonality: If $p_1, \ldots, p_n$ are distinct primes

$$\sum_{t(p_1 \cdots p_n)} a_{t_1}^{r_1}(p_1) \cdots a_{t_n}^{r_n}(p_n) = A_{r_1,\mathcal{F}}(p_1) \cdots A_{r_n,\mathcal{F}}(p_n).$$

Needed Input

For many families

$$\begin{aligned}(1) : A_{1,\mathcal{F}}(p) &= -rp + O(1)\\ (2) : A_{2,\mathcal{F}}(p) &= p^2 + O(p^{3/2})\end{aligned}$$

Rational Elliptic Surfaces (Silverman and Rosen):

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -A_{\mathcal{E}}(p)\log p = r$$

Surfaces with $j(t)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

New Results

Rational Surfaces Density Theorem: *Consider a* 1-*parameter family of elliptic curves of rank* r *over* $\mathbb{Q}(t)$ *that is a rational surface. Assume GRH,* $j(t)$ *non-constant, and the ABC (or Sq-Free Sieve) conjecture if* $\Delta(t)$ *has an irreducible polynomial factor of degree* ≥ 4*. Let* $m =$ *deg* $C(t)$ *and* f_i *be an even Schwartz function of small support* σ_i *(*$\sigma_1 < \min(\frac{1}{2}, \frac{2}{3m})$ *for the* 1*-level density,* $\sigma_1 + \sigma_2 < \frac{1}{3m}$ *for the* 2*-level density). Possibly after passing to a subsequence, we observe two pieces. The first equals the expected contribution from* r *zeros at the critical point (agreeing with what B-SD suggests). The second is*

$$\begin{aligned} D_{1,\mathcal{F}}^{(r)}(f_1) &= \widehat{f_1}(0) + \frac{1}{2}f_1(0) \\ D_{2,\mathcal{F}}^{(r)}(f) &= \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du \\ &\quad -2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1f_2)(0)N(\mathcal{F},-1) \end{aligned}$$

where $N(\mathcal{F},-1)$ *is the percent of curves with odd sign.*

1 and 2-level densities confirm Katz-Sarnak predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free: all even.
2. $y^2 = x^3 \pm 4(4t+2)x$, $4t+2$ Sq-Free: $+$ yields all odd, $-$ yields all even.
3. $y^2 = x^3 + tx^2 - (t+3)x + 1$, t^2+3t+9 Sq-Free: all odd.

First two rank 0 over $\mathbb{Q}(t)$; third is rank 1. Only assume GRH for first two; add B-SD to interpret third.

Family of Rank 6 over $\mathbb{Q}(t)$ (modulo reasonable conjs):

$$\begin{aligned} y^2 &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ &\quad +(2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8916100448256000000 \\ B &= -811365140824616222208 \\ C &= 26497490347321493520384 \\ D &= -343107594345448813363200 \\ a &= 16660111104 \\ b &= -1603174809600 \\ c &= 2149908480000 \end{aligned}$$

Sieving

If conductors were constant and summed over $t \in [N, 2N]$, would have $\frac{N}{p_1 p_2}$ complete sums, each giving $\prod_i A_{r_i,\mathcal{F}}(p_i)$.

Let $D(t)$ be the product of the irreducible factors of $\Delta(t)$. Can often show $C(t)$ is a monotone polynomial of t when $D(t)$ is square-free, and there are $c_{\mathcal{F}} N + o(N)$ such t. (Unconditional if all factors of $D(t)$ of degree ≤ 3; else need ABC or Square-Free Sieve conjecture).

$$\begin{aligned}
\sum_{\substack{t=N \\ D(t) \\ sqfree}}^{2N} S(t) &= \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t) \\
&= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t) + \sum_{d\geq \log^l N}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t).
\end{aligned}$$

Handle first piece by progressions, handle second piece by Cauchy-Schwartz.

Sieving (cont)

The number of t in the second sum is $o(N)$ (unconditionally if all of $D(t)$'s factors are deg ≤ 3). Denote these t by $\mathcal{T}$. By Cauchy-Schwartz:

$$\begin{aligned}\sum_{t\in\mathcal{T}} S(t) &\ll \left(\sum_{t\in\mathcal{T}} S^2(t)\right)^{\frac{1}{2}} \cdot \left(\sum_{t\in\mathcal{T}} 1\right)^{\frac{1}{2}} \\ &\ll \left(\sum_{t\in[N,2N]} S^2(t)\right)^{\frac{1}{2}} \cdot o\left(\sqrt{N}\right).\end{aligned}$$

Done if $\sum_{t=N}^{2N} S^2(t) = O(N)$.

First piece is handled by progressions: let $\nu(d)$ be the number of incongruent roots of $D(t) \equiv 0 \bmod d^2$; $\nu(d) \ll d^\epsilon$. Let $t_i(d)$ be one of the $\nu(d)$ roots. This gives a sequence of t: $t_i(d), t_i(d)+d^2, \dots, t_i(d)+[\frac{N}{d^2}]d^2$.

If $(d, p_1p_2) = 1$, then (mod p_1p_2), go through the complete set of residue classes $\frac{N/d^2}{p_1p_2}$ times. As $d < \log^l N$, $l < 2$, can take all $p_i > \log^l N$ in the Explicit Formula, incorporating lower p_i's into the error terms.

Partial Summation

Notation: $\widetilde{a}_{d,i,p}(t') = a_{t(d,i,t')}(p)$, $G_{d,i,P}(u)$ is related to the test functions, d and i from progressions.

Applying Partial Summation

$$\begin{aligned} S(d,i,r,p) &= \sum_{t'=0}^{[N/d^2]} \widetilde{a}^{r}_{d,i,p}(t') G_{d,i,p}(t') \\ &= \left(\frac{[N/d^2]}{p} A_{r,\mathcal{F}}(p) + O\Big(p^R\Big) \right) G_{d,i,p}([N/d^2]) \\ &\quad - \sum_{u=0}^{[N/d^2]-1} \left(\frac{u}{p} A_{r,\mathcal{F}}(p) + O\Big(p^R\Big) \right) \\ &\quad \cdot \left(G_{d,i,p}(u) - G_{d,i,p}(u+1) \right) \end{aligned}$$

$O(p^R)$ is the error from using Hasse to bound the partial sums: $p^R = p^{1+\frac{r}{2}}$.

Difficult Piece: Fourth Sum I

$$\sum_{u=0}^{[N/d^2]-1} O(P^R)\Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big)$$

Taylor Expansion for $G_{d,i,P}(u) - G_{d,i,P}(u+1)$ is not sufficient. Gives $\frac{|\mathcal{F}|P^R}{d^2 \log N}$, and

$$\frac{1}{|\mathcal{F}|}\sum_{d,i}\frac{|\mathcal{F}|P^R}{d^2\log N} \ = \ O\left(\frac{P^R}{\log N}\right). \quad (0.0.1)$$

The problem is in summing over the primes, as we no longer have $\frac{1}{|\mathcal{F}|}$. We multiply by $\frac{1}{P^r}$.

Consider the case $r = (1,0)$. Then $P = p_1 = p$, $R = 1 + \frac{r_1}{2} = \frac{3}{2}$, and $\frac{1}{P^r} = \frac{1}{p}$. We have

$$\sum_{p=\log^l N}^{N^{m\sigma}} \frac{1}{p}\frac{p^{\frac{3}{2}}}{\log N} \gg N^{m\sigma}.$$

As $N \to \infty$, this term blows up. We need much better cancellation. Note, by Hasse, $O(P^R) \leq 2^R P^R$.

Fourth Sum: II

If exactly one of the r_j's is non-zero, then

$$\sum_{u=0}^{[N/d^2]-1} \left| G_{d,i,P}(u) - G_{d,i,P}(u+1) \right|$$
$$= \sum_{u=0}^{[N/d^2]-1} \left| g\left(\frac{\log p}{\log C(t_i(d)+ud^2)}\right) - g\left(\frac{\log p}{\log C(t_i(d)+(u+1)d^2)}\right) \right|$$

If the conductors are monotone, for fixed i, d and p, by MVT is small. Essential that we have a problem in bounded variation of g and not $g_{d,i,P}$.

g has bounded derivative bounds the u-sum by the support of g.

If two of the r_j's are non-zero, use the following:

$$\begin{aligned} |a_1a_2 - b_1b_2| &= |a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2| \\ &\leq |a_1a_2 - b_1a_2| + |b_1a_2 - b_1b_2| \\ &= |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2| \end{aligned}$$

Note: *if our conductors are not monotone, we cannot apply bounded variation; we could transverse* $[0, 1000\sigma]$ *(or a large subset of it) many times.*

Handling the Conductors

$$
\begin{aligned}
C(t) &= \prod_{p|\Delta(t)} p^{f_p(t)} \\
D_1(t) &= \text{primitive irred. poly. factors } \Delta(t) \text{ and } c_4(t) \text{ share} \\
D_2(t) &= \text{remaining primitive irred. poly. factors of } \Delta(t) \\
D(t) &= D_1(t)D_2(t)
\end{aligned}
$$

$D(t)$ square-free, $C(t)$ like $D_1^2(t)D_2(t)$ except for a finite set of bad primes.

Let P be the product of the bad primes.

By Tate's Algorithm, can determine $f_p(t)$, which depends on the coefficients $a_i(t)$ mod powers of p.

Apply Tate's Algorithm to E_{t_1} to determine $f_p(t_1)$ for the bad primes. m large, $f_p(\tau) = f_p(P^m t + t_1) = f_p(t_1)$ for $p|P$.

m enormous, for bad primes, the order of p dividing $D(P^m t + t_1)$ is independent of t. So can find integers st $C(\tau) = c_{bad}\frac{D_1^2(\tau)}{c_1}\frac{D_2(\tau)}{c_2}$, $D(\tau)$ square-free.

Application: Bounding Excess Rank

$$D_{1,\mathcal{F}}(f_1) = \widehat{f_1}(0) + \frac{1}{2}f_1(0) + rf_1(0).$$

To estimate the percent with rank at least $r + R$, P_R, we get

$$Rf_1(0)P_R \leq \widehat{f_1}(0) + \frac{1}{2}f_1(0), \ R > 1.$$

Note the family rank r has been cancelled from both sides.

By using the 2-level density, however, we get *squares* of the rank on the left hand side. The advantage is we get a cross term rR. The disadvantage is our support is smaller. Once R is large, the 2-level density yields better results.

Potential Lower Order Density Terms

Can often show

$$A_{2,\mathcal{F}}(p) = p^2 - m_{\mathcal{F}} \cdot p + O(1),\ m_{\mathcal{F}} > 0.$$

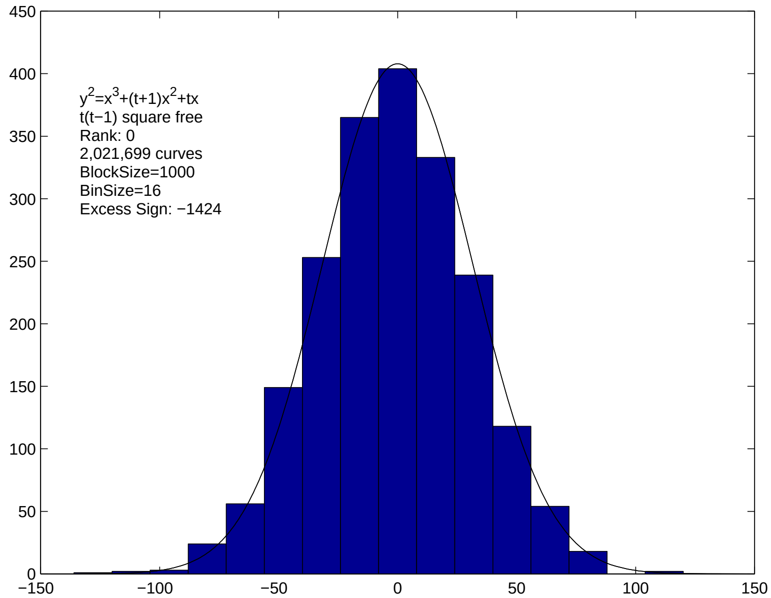
The p^2 term contributes $-\frac{1}{2}f(0)$; the $m_{\mathcal{F}}$ term contributes something of size $\frac{1}{\log N}$.

The potential lower order density terms, arising from lower order terms in the sums of the second moments of $a_E^2(p)$, could be masked by the errors propagating through our derivations. We have errors of the size $\frac{\log\log N}{\log N}$ arising from the Explicit Formula and the contributions from $a_E^m(p)$, $m \geq 3$.

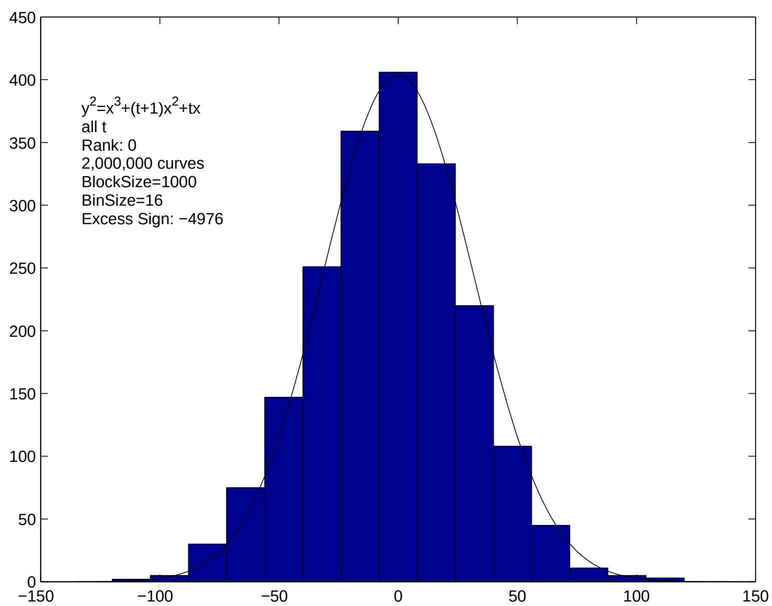
To truly observe lower order corrections to the densities, a significantly more delicate analysis of these discarded terms are needed. The conductor dependence in the Gamma factors of the Explicit Formula are easily managed. The real difficulty is handling the primes which divide the discriminant and the $m \geq 3$ terms.

We save this for a future project, and content ourselves with observing a potential lower order density term.

Distribution of Signs: $y^2 = x^3 + (t+1)x^2 + tx$

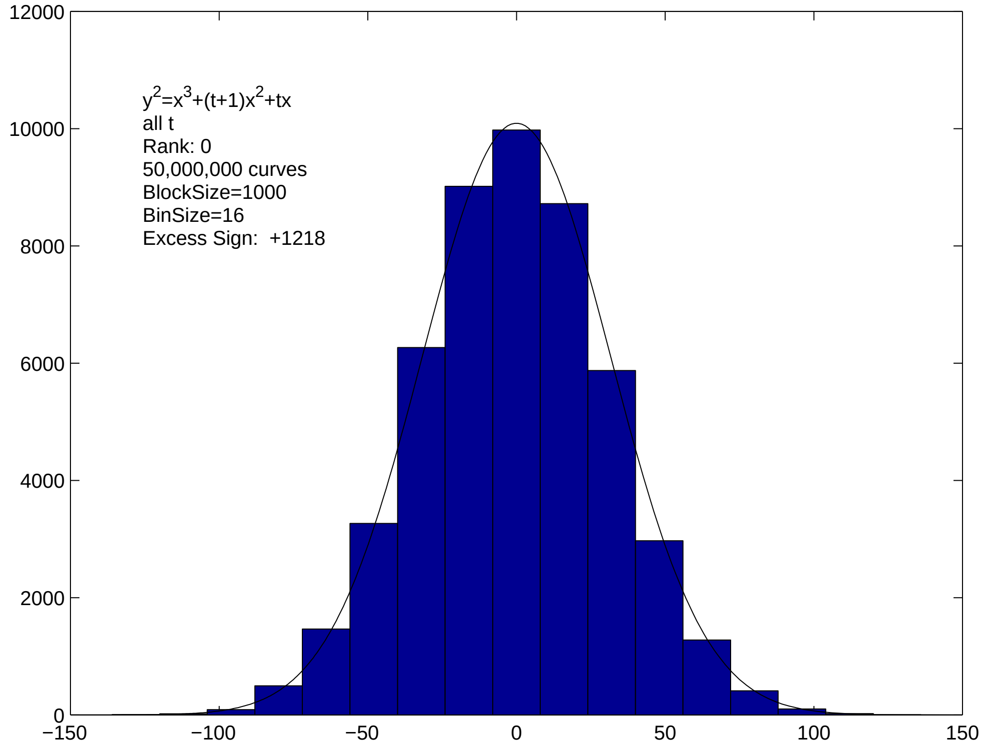


Histogram plot: $D(t)$ sq-free, first $2 \cdot 10^6$ such t.



Histogram plot: All $t \in [2, 2 \cdot 10^6]$.

Distribution of signs: $y^2 = x^3 + (t+1)x^2 + tx$



Histogram plot: All $t \in [2, 5 \cdot 10^7]$

The observed behavior agrees with the predicted behavior. Note as the number of curves increase (comparing the plot of $5 \cdot 10^7$ points to $2 \cdot 10^6$ points), the fit to the Gaussian improves.

Graphs by Atul Pokharel