To Infinity and Beyond: Gaps Between Summands in Zeckendorf Decompositions

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Introduction

Goals of the Talk

- Review previous work on Zeckendorf-type decompositions.
- Describe new results on gaps between summands.
- Discuss open problems being studied by SMALL 2012.

Thanks to colleagues from the Williams College 2010 and 2011 SMALL REU programs (especially Murat Kologlu, Gene Kopp and Yinghui Wang).

Previous Results

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Example:
$2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1$.

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1},\ n \geq L$$

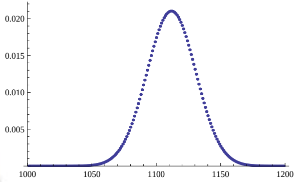
with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1,\ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker: The average number of summands in the generalized Zeckendorf decomposition for integers in $[H_n, H_{n+1})$ tends to $Cn + d$ as $n \to \infty$, where $C > 0$ and d are computable constants determined by the c_i's.
- Central Limit Type Theorem

Central Limit Type Theorem

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$, is Gaussian.



Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \ldots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \ldots, 9\}$.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.
- For large n, the contribution of A_n is immaterial. A_i $(1 \leq i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.

Preliminaries: The Cookie Problem

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Proof: Consider $C + P - 1$ cookies in a line.
Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.
Divides the cookies into P sets.

The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

Cookie counting $\Rightarrow$ $p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.

New Approach: Case of Fibonacci Numbers

$p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:
  $N \in [F_{n+1}, F_{n+2})$: $N = F_{n+1} + F_t + \cdots$, $t \leq n-1$.

$$\begin{aligned} p_{n+1,k+1} &= p_{n-1,k} + p_{n-2,k} + \cdots \\ p_{n,k+1} &= p_{n-2,k} + p_{n-3,k} + \cdots \\ \Rightarrow p_{n+1,k+1} &= p_{n,k+1} + p_{n-1,k}. \end{aligned}$$

- Generating function: $\sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1-y-xy^2}$.
- Partial fraction expansion:

$$\frac{y}{1-y-xy^2} = -\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)} - \frac{1}{y-y_2(x)}\right)$$

where $y_1(x)$ and $y_2(x)$ are the roots of $1-y-xy^2=0$.

Coefficient of y^n: $g(x) = \sum_{k>0} p_{n,k} x^k$.

New Approach: Case of Fibonacci Numbers (Continued)

K_n: the corresponding random variable associated with k.
$g(x) = \sum_{k>0} p_{n,k} x^k$.

- Differentiating identities:
  $g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n$,
  $g'(x) = \sum_{k>0} k p_{n,k} x^{k-1}$, $g'(1) = g(1) E[K_n]$,
  $(xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1}$,
  $(xg'(x))' |_{x=1} = g(1) E[K_n^2]$, $\left(x\,(xg'(x))'\right)' |_{x=1} = g(1) E[K_n^3]$, ...
  Similar results hold for the centralized K_n: $K_n' = K_n - E[K_n]$.
- Method of moments (for normalized K_n'):
  $E[(K_n')^{2m}]/(SD(K_n'))^{2m} \to (2m-1)!!$,
  $E[(K_n')^{2m-1}]/(SD(K_n'))^{2m-1} \to 0$. $\Rightarrow K_n \to$ Gaussian.

New Approach: General Case

Let $p_{n,k} = \#\{N \in [H_n, H_{n+1})$: the generalized Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:

  Fibonacci: $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$.

  General: $p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}$.

  where $s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m$.

- Generating function:

  Fibonacci: $\frac{y}{1-y-xy^2}$.

  General:

$$\frac{\sum_{n \le L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k} x^k y^n}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.$$

New Approach: General Case (Continued)

- Partial fraction expansion:

Fibonacci: $-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)}-\frac{1}{y-y_2(x)}\right)$.

General:

$$-\frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j}\sum_{i=1}^{L}\frac{B(x,y)}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))}.$$

$$B(x,y)=\sum_{n\leq L} p_{n,k}x^k y^n-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}\sum_{n<L-m} p_{n,k}x^k y^n,$$

$$y_i(x)\text{: root of } 1-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}=0.$$

Coefficient of y^n: $g(x)=\sum_{n,k>0} p_{n,k}x^k$.

- Differentiating identities
- Method of moments $\Rightarrow K_n \to$ Gaussian

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Gaps Between Summands

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

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Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Results (Beckwith-Miller 2011)

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Fibonacci Results

Theorem (Zeckendorf Gap Distribution (BM))

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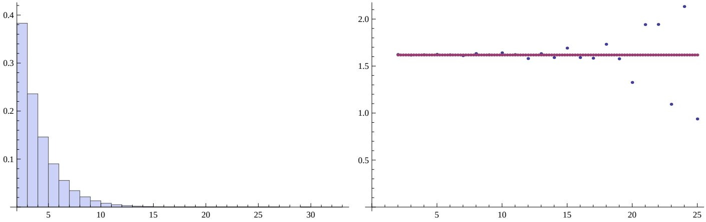

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{1000} \approx 10^{208}$.

Main Results (Gaudet-Miller 2012)

Generalized Fibonacci Numbers: $G_n = G_{n-1} + \cdots + G_{n-L}$.

Theorem (Gaps for Generalized Fibonacci Numbers)

The limiting probability of finding a gap of length $k \geq 1$ between summands of numbers in $[G_n, G_{n+1}]$ decays geometrically in k:

$$P(k) \;=\; \begin{cases} \dfrac{p_1(\lambda_{1;L}^2 - \lambda_{1;L} - 1)^2}{C_L}\,\lambda_{1;L}^{-1} & \text{if } k = 1 \\[2ex] \dfrac{p_1(\lambda_{1;L}^{L-1} - 1)}{C_L \lambda_{1;L}^{L-1}}\,\lambda_{1;L}^{-k} & \text{if } k \geq 2, \end{cases}$$

where $\lambda_{1;L}$ is the largest eigenvalue of the characteristic polynomial, $G_n = p_1\lambda_{1;L}^n + \cdots$ and C_L is a constant.

Gap Proofs

Proof of Fibonacci Result

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

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For the indices less than i: F_{i-1} choices. Why? Have F_i, don't have F_{i-1}. Follows by Zeckendorf: like the interval $[F_i, F_{i+1})$ as have F_i, number elements is $F_{i+1} - F_i = F_{i-1}$.

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For the indices greater than $i+k$: $F_{n-k-i-2}$ choices. Why? Have F_n, don't have F_{i+k+1}. Like Zeckendorf with potential summands $F_{i+k+2}, \dots, F_n$. Shifting, like summands $F_1, \dots, F_{n-k-i-1}$, giving $F_{n-k-i-2}$.

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For the indices greater than $i+k$: $F_{n-k-i-2}$ choices. Why? Have F_n, don't have F_{i+k+1}. Like Zeckendorf with potential summands $F_{i+k+2}, \dots, F_n$. Shifting, like summands $F_1, \dots, F_{n-k-i-1}$, giving $F_{n-k-i-2}$.

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

$$\sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$

- $\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $(g(x))^2$, where $g(x)$ is the generating function of the Fibonaccis.
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- Alternatively, use Binet's formula and get sums of geometric series.

$P(k) = C/\phi^k$ for some constant C, so $P(k) = \phi(\phi - 1)/\phi^k$.

Tribonacci Gaps

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Counting:

$$X_{i,i+k}(n) = \begin{cases} T_{i-1}(T_{n-i-3} + T_{n-i-4}) & \text{if } k = 1 \\ (T_{i-1} + T_{i-2})(T_{n-k-i-1} + T_{n-k-i-3}) & \text{if } k \geq 2. \end{cases}$$

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Constants s.t. $P(1) = \frac{c_1}{C\lambda_1^3}$, $P(k) = \frac{2c_1}{C(1+\lambda_1)}\lambda_1^{-k}$ (for $k \geq 2$).

Similar argument works as all coefficients are 1.

Future Work

Other gaps?

- Gaps longer than recurrence – proved geometric decay.
- Interesting behavior with “short” gaps.
- “Skiponaccis”: $S_{n+1} = S_n + S_{n-2}$.
- “Doublanaccis”: $H_{n+1} = 2H_n + H_{n-1}$.
- Hope: Generalize to all positive linear recurrences.

Thank you!