

Pairwise relative primality of positive integers

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Open Problems

The following open problems are raised by P.

Moree:

- To compute the probability that k positive integers have at least r relatively prime pairs.
- To compute the probability that k positive integers have exactly r relatively prime pairs.
- To compute the probability that k positive integers are pairwise not relatively prime.

False answer

The probability that a pair of positive integers are relatively prime = $\frac{1}{\zeta(2)}$

The probability that k positive integers have exactly r relatively prime pairs

$$= \left(\frac{1}{\zeta(2)} \right)^r \left(1 - \frac{1}{\zeta(2)} \right)^{\binom{k}{2} - r}$$

It is false, even when k=3.

Classical results

Results of Lehmer (1900) and Nymann (1972):

The probability that k positive integers a_1, \dots, a_k are relatively prime is

$$\prod_p (1 - 1/p^k) = 1/\zeta(k)$$

Classical results

Outline of proof:

Define

$Q(n)$ = the number of k -tuples of positive integers less than or equal to n that are relatively prime.

Then by the Inclusion-Exclusion Principle,

Classical results

$$\begin{aligned} Q(n) &= \sum_{j=1}^n \mu(j) \left\lfloor \frac{n}{j} \right\rfloor^k \\ &= \sum_{j=1}^n \mu(j) \left(\frac{n}{j}\right)^k + O\left(\sum_{j=1}^n \left(\frac{n}{j}\right)^{k-1}\right) \\ &= \frac{n^k}{\zeta(k)} + \begin{cases} O(n \log n), & k = 2, \\ O(n^{k-1}), & k > 2. \end{cases} \end{aligned}$$

Toth's work

Results of Toth (2002):

The probability that k positive integers are pairwise relatively prime is

$$\prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right)$$

Toth's work

Define

$Q_k^{(u)}(n)$ = the number of k -tuples a_1, \dots, a_k with $1 \leq a_1, \dots, a_k \leq n$ such that a_1, \dots, a_k are pairwise relatively prime and each is prime to u .

Toth's work

Then

$$Q_k^{(u)}(n) = A_k f_k(u) n^k + O(\theta(u) n^{k-1} \log^{k-1} n),$$

where

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 + \frac{k-1}{p}\right),$$

$$f_k(u) = \prod_{p|u} \left(1 - \frac{k}{p+k-1}\right),$$

and $\theta(u)$ is the number of square free divisors of u .

The general case: k-wise relatively prime

Let $A_{s,k}$ denote the probability that s positive integers are k -wise relatively prime.

Obviously,

$$A_{s,2} < A_{s,3} < A_{s,4} < \dots < A_{s,s} = 1/\zeta(s)$$

The general case: k-wise relatively prime

For a $(k-1)$ -tuple of positive integers $u=(u_1, u_2, \dots, u_{k-1})$ such that $(u_i, u_j)=1$ for $i \neq j$, we denote

$Q_{s,k}^{(u)}(n)$ the number of s -tuples a_1, \dots, a_s with $1 \leq a_1, \dots, a_s \leq n$ such that they are k -wise relatively prime and i -wise relatively prime to u_i for $i=1, 2, \dots, k-1$.

The general case: k-wise relatively prime

$$Q_{s,k}^{(u)}(n) = A_{s,k} \prod_{i=1}^{k-1} f_{s,k,i}(u_i) n^s + O(\theta(u_1) n^{s-1} \log^{\delta(s,k)} n)$$

where

$$A_{s,k} = \prod_p \left(1 - \frac{1}{p}\right)^{s-k+1} \left(\sum_{m=0}^{k-1} \binom{s}{m} \left(1 - \frac{1}{p}\right)^{k-1-m} \left(\frac{1}{p}\right)^m\right),$$

$$f_{s,k,i}(u) = \prod_{p|u_i} \left(1 - \frac{\sum_{m=i}^{k-1} \binom{s}{m} (p-1)^{k-1-m}}{\sum_{m=0}^{k-1} \binom{s}{m} (p-1)^{k-1-m}}\right),$$

$$\delta(s,k) = \max \left\{ \binom{s-1}{i}, i = 1, 2, \dots, k-1 \right\}.$$

The general case: k-wise relatively prime

The probability that s positive integers are
k-wise relatively prime = $A_{s,k}$

$$= \prod_p \left(1 - \frac{1}{p}\right)^{s-k+1} \left(\sum_{m=0}^{k-1} \binom{s}{m} \left(\frac{1}{p}\right)^m \left(1 - \frac{1}{p}\right)^{k-1-m} \right)$$

The general case: k-wise relatively prime

The recurrence relation:

$$Q_{s+1,k}^{(u)}(n) = \sum_{\substack{j=1 \\ (j,u_1)=1}}^n Q_{s,k}^{(j^*u)}(n)$$

where

$$j^*u = \left(\frac{u_1(j, u_2)}{(j, u_1]}, \frac{u_2(j, u_3)}{(j, u_2]}, \dots, \frac{u_{k-2}(j, u_{k-1})}{(j, u_{k-2}]}, \frac{ju_{k-1}}{\prod_{i=1}^{k-1} [(j, u_i)(j, u_{k-1})]} \right)$$

Moore and Freiberg's work

A pair of positive integers (a, b) is carefree if a is both squarefree and relatively prime to b .
If b is also squarefree, we say that (a, b) is a strongly carefree couple.

Moree and Freiberg's work

Moree's result:

K_1 = the probability that a pair of positive integers are carefree

$$= \frac{1}{\zeta(2)} \prod_p \left(1 - \frac{1}{p(p+1)}\right)$$

K_2 = the probability that a pair of positive integers are strongly carefree

$$= \frac{1}{\zeta(2)^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right)$$

Moree and Freiberg's work

Freiberg's result (based on Moree's work):

F_3 = the probability that three positive integers are pairwise not relatively prime

$$= 1 - \frac{3}{\zeta(2)} + 3K_1 - K_2$$

G-wise relatively prime

$G=(V,E)$ with $V=\{1,2,3,\dots,k\}$.

A k -tuple of positive integers a_1, a_2, \dots, a_k are G -wise relatively prime if $(a_i, a_j)=1$ for edge $\{i,j\}$ in E .

For a k -tuple positive integers $u=(u_1, u_2, \dots, u_k)$, let $Q_G^{(u)}(n)$ = the number k -tuples of positive integers a_1, a_2, \dots, a_k with $1 \leq a_1, a_2, \dots, a_k \leq n$ such that $(a_i, u_i)=1$ for $i=1, 2, \dots, k$ and they are G -wise relatively prime.

G-wise relatively prime

Let $i_m(G)$ denote the number of independent sets of cardinality m .

For a subset S of V , let $i_m(G,S)$ denote the number of independent sets of cardinality m with at least one vertex in S .

Let $V(u,p)=\{i : p \mid u_i\}$, $i_m(G,V(u,p))=$ the number of independent sets of cardinality m with at least one vertex i such that $p \mid u_i$.

G-wise relatively prime

$$Q_G^{(u)}(n) = A_G f_G(u) n^k + O(\theta(u) n^{k-1} \log^{k-1} n)$$

where

$$A_G = \prod_p A_{G,p} = \prod_p \left(\sum_{m=0}^k i_m(G) \left(1 - \frac{1}{p}\right)^{k-m} \left(\frac{1}{p}\right)^m \right),$$

$$f_G(u) = \prod_{p|u_1 u_2 \cdots u_k} \left(1 - \frac{\sum_{m=0}^k i_m(G, V(u, p)) (p-1)^{k-m}}{\sum_{m=0}^k i_m(G) (p-1)^{k-m}} \right),$$

$$\theta(u) = \max\{\theta(u_i), i = 1, 2, \dots, k\}.$$

G-wise relative prime

$$A_{G,p} = \left(1 - \frac{1}{p}\right) A_{G-v,p} + \left(1 - \frac{1}{p}\right)^{\deg v} \frac{1}{p} A_{G-N[v],p}$$

where v is a vertex of G and $N[v]$ is the closed neighborhood of v .

Pairwise not relatively prime

Let $A_{k,\geq r}$ denote the probability that k positive integers have at least r relatively prime pairs.

Let $A_{k,=r}$ denote the probability that k positive integers have exactly r relatively prime pairs.

Pairwise not relatively prime

$$A_{k,\geq r} = \sum_{i=r}^{k(k-1)/2} (-1)^{i-r} \binom{i-1}{r-1} B_{k,i}$$

$$A_{k,=r} = \sum_{i=r}^{k(k-1)/2} (-1)^{i-r} \binom{i}{r} B_{k,i}$$

where

$$B_{k,i} = \sum_{|E(G)|=i} A_G$$

Pairwise not relatively prime

$A_{k,=0}$ = the probability that k positive integers are pairwise not relatively prime

$$= \sum_{i=0}^{k(k-1)/2} (-1)^i B_{k,i}$$

Recurrence relation

For a graph $G=(V,E)$ with $V=\{1,2,\dots,k+1\}$ and a $(k+1)$ -tuple positive integers $u=\{u_1,u_2,\dots,u_{k+1}\}$,

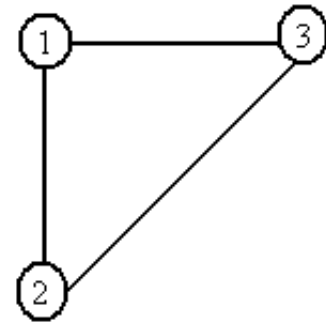
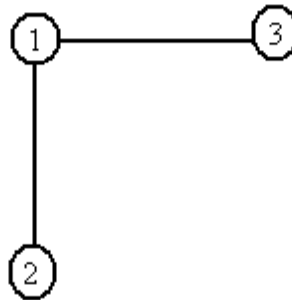
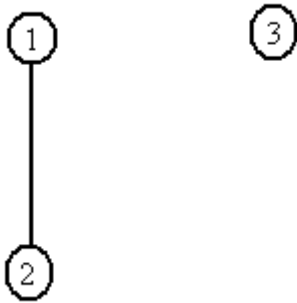
$$Q_G^{(u)}(n) = \sum_{\substack{j=1 \\ (j,u_{k+1})=1}}^n Q_{G-v}^{(j*u)}(n)$$

where $v=\{k+1\}$ and if $(j*u)_i$ denotes the i th component of $j*u$, then

$$(j*u)_i = \begin{cases} ju_i & \{i, k+1\} \in E \\ u_i & \text{otherwise} \end{cases}$$

$$k=3$$

For $G=(V,E)$ with $V=\{1,2,3\}$, $i_0(G)=1$, $i_1(G)=3$, and if $|E|>0$, $i_3(G)=0$.



$$i_2(G)=2$$

$$A_{3,1} = \frac{1}{\zeta(2)}$$

$$i_2(G)=1$$

$$A_{3,2} = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \left(1 - \frac{1}{p}\right) \frac{1}{p}\right) = K_1$$

$$i_2(G)=0$$

$$A_{3,3} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) = K_2$$

$$k=3$$

$$B_{3,0}=1, B_{3,1}=\frac{3}{\zeta(2)}, B_{3,2}=3K_1, B_{3,3}=K_2$$

The probability that 3 positive integers are relatively prime = $A_{3,0}$

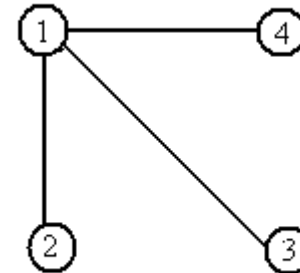
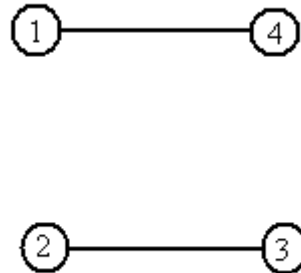
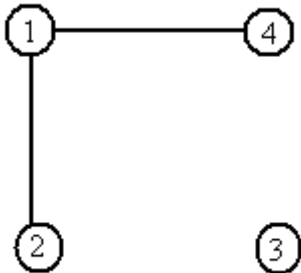
$$=B_{3,0}-B_{3,1}+B_{3,2}-B_{3,3}$$

$$=1-\frac{3}{\zeta(2)}+3K_1-K_2$$

We recover Freiberger's result.

$$k=4$$

For a graph $G=(V,E)$ with $V=\{1,2,3,4\}$, $i_0(G)=1$,
 $i_1(G)=4$, and if $|E|>0$, $i_4(G)=0$.



$$i_2(G)=4, i_3(G)=1$$

of graphs=12

$$A_{4,2,1} = A_{3,2}$$

$$i_2(G)=4, i_3(G)=0$$

of graphs=3

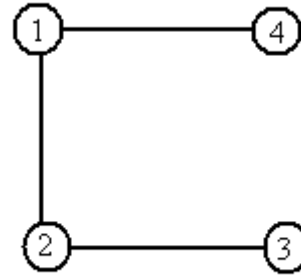
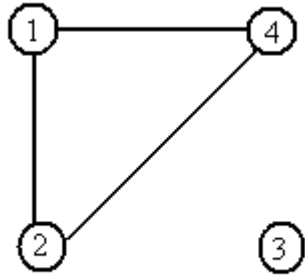
$$A_{4,2,2} = \frac{1}{\zeta(2)^2}$$

$$i_2(G)=3, i_3(G)=1$$

of graphs=4

$$A_{4,3,1} = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \left(1 - \frac{1}{p}\right)^2 \frac{1}{p}\right)$$

$k=4$



of graphs = 4

$$i_2(G)=3, i_3(G)=0, \\ =K_2$$

of graphs = 12

$$A_{4,3,2} = A_{3,3} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right)$$

$|E|=4$, # of graphs=15, $i_2(G)=2$, $i_3(G)=0$,

$$A_{4,4} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \left(2 - \frac{1}{p}\right) \frac{1}{p}\right)$$

$$k=4$$

$|E|=5$, # of graphs = 6, $i_2(G)=1$, $i_3(G)=0$

$$A_{4,5} = \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + 2\left(1 - \frac{1}{p}\right) \frac{1}{p}\right)$$

$|E|=6$, # of graphs = 1, $i_2(G)=0$, $i_3(G)=0$

$$A_{4,6} = \prod_p \left(1 - \frac{1}{p}\right)^3 \left(1 + \frac{3}{p}\right)$$

$$B_{4,0}=1, B_{4,1}=6A_{4,1} = \frac{6}{\zeta(2)}$$

$$B_{4,2}=12A_{4,2,1}+3A_{4,2,2}=12K_1 + \frac{3}{\zeta(2)^2}$$

$$B_{4,3}=4A_{4,3,1}+16A_{4,3,2}=4A_{4,3,1}+16K_2$$

$$k=4$$

$$B_{4,4}=15A_{4,4}, \quad B_{4,5}=6A_{4,5}, \quad B_{4,6}=A_{4,6} .$$

The probability that 4 positive integers are
pairwise not relatively prime = $A_{4,=0}$

$$= 1 - 6A_{4,1} + 12A_{4,2,1} + 3A_{4,2,2} - 4A_{4,3,1} - 16A_{4,3,2} + 15A_{4,4} \\ - 6A_{4,5} + A_{4,6}$$



Thank you!