The Distribution of Generalized Ramanujan Primes

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Combinatorial and Additive Number Theory (CANT 2012)
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- For fixed $c \in (0, 1)$, we expect the amount of primes in an interval $(cx, x]$ to increase with x.

Historical Introduction

Bertrand's Postulate (1845)

For all integers $x \geq 2$, there exists at least one prime in $(x/2, x]$.

Ramanujan Primes

Definition

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- Sondow: As $x \to \infty$, 50% of primes $\leq x$ are Ramanujan.

c-Ramanujan Primes

Definition

For $c \in (0, 1)$, the n-th c-Ramanujan prime $R_{c,n}$ is the smallest integer such that for any $x \geq R_{c,n}$, at least n primes are in $(cx, x]$.

Preliminaries

Let $\pi(x)$ be the prime-counting function that gives the number of primes less than or equal to x.

The Prime Number Theorem states:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1.$$

Preliminaries

The logarithmic integral function $\mathrm{Li}(x)$ is defined by

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t} dt.$$

The Prime Number Theorem gives us

$$\pi(x) = \mathrm{Li}(x) \; + \; O\left(\frac{x}{\log^2 x}\right),$$

i.e., there is a $C > 0$ such that for all x sufficiently large

$$-C\frac{x}{\log^2 x} \; \leq \; \pi(x) - \mathrm{Li}(x) \; \leq \; C\frac{x}{\log^2 x}.$$

Existence of $R_{c,n}$

Theorem (ABMRS 2011)

For all $n \in \mathbb{Z}$ and all $c \in (0, 1)$, the n-th c-Ramanujan prime $R_{c,n}$ exists.

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- The number of primes in $(cx, x]$ is $\pi(x) - \pi(cx)$.
- Using the Prime Number Theorem and Mean Value Theorem, there exists a $b_c \in [0, -\log c]$,

$$\pi(x) - \pi(cx) \;=\; \frac{(1-c)x}{\log x - b_c} + O\left(\frac{x}{\log^2 x}\right).$$

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- For any integer n and for all x sufficiently large, $\pi(x) - \pi(cx) \geq n$.

Asymptotic Behavior

Theorem (ABMRS 2011)

For any fixed $c \in (0, 1)$, the n-th c-Ramanujan prime is asymptotic to the $\frac{n}{1-c}$-th prime as $n \to \infty$.

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By the triangle inequality

$$\begin{aligned} \left|R_{c,n} - p_{\frac{n}{1-c}}\right| \quad &\leq \quad \left|R_{c,n} - \frac{n}{1-c}\log R_{c,n}\right| + \left|\frac{n}{1-c}\log R_{c,n} - \frac{n}{1-c}\log n\right| \\ &+ \left|\frac{n}{1-c}\log n - \frac{n}{1-c}\log\frac{n}{1-c}\right| \\ &+ \left|\frac{n}{1-c}\log n - p_{\frac{n}{1-c}}\right| \end{aligned}$$

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Since $\frac{n\log\log n}{p_{\frac{n}{1-c}}} \to 0$ as $n \to \infty \Rightarrow R_{c,n} \sim p_{\frac{n}{1-c}}$.

Frequency of c-Ramanujan Primes

Theorem (ABMRS 2011)

In the limit, the probability of a generic prime being a c-Ramanujan prime is $1 - c$.

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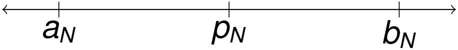

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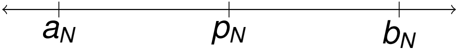

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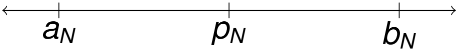

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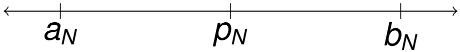

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  - $\mathrm{R}_{c,n} = a_N$ and every prime in $(a_N, p_N]$ is c-Ramanujan,
  - $\mathrm{R}_{c,n} = b_N$ and every prime in $[p_N, b_N)$ is c-Ramanujan.

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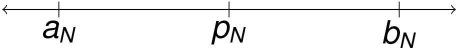

- Worst cases:
  - $R_{c,n} = a_N$ and every prime in $(a_N, p_N]$ is c-Ramanujan,
  - $R_{c,n} = b_N$ and every prime in $[p_N, b_N)$ is c-Ramanujan.
- Goal: $\frac{\pi(b_N) - \pi(a_N)}{\pi(p_N)} \to 0$ as $N \to \infty$.

Frequency of c-Ramanujan Primes

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$$a_N = p_N - \beta_c N \log\log N, \quad b_N = p_N + \beta_c N \log\log N.$$

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- Then, $R_{c,n} \in [a_N, b_N]$.
- Using the Prime Number Theorem, we can show

$$\frac{\pi(b_N) - \pi(a_N)}{\pi(p_N)} \leq \xi_c \frac{\log\log N}{\log N} \to 0 \text{ as } N \to \infty.$$

Prime Numbers

2	3	5	7	11	13	17
19	23	29	31	37	41	43
47	53	59	61	67	71	73
79	83	89	97	101	103	107
109	113	127	131	137	139	149
151	157	163	167	173	179	181
191	193	197	199	211	223	227

Ramanujan Primes ($c = \frac{1}{2}$)

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$$\begin{aligned}\mathbb{E}[L_N] &\approx \frac{\log N}{\log(1/P)} - \left(\frac{1}{2} - \frac{\log(1-P)+\gamma}{\log(1/P)}\right), \\ \mathrm{Var}[L_N] &\approx \frac{\pi^2}{6\log^2(1/P)} + \frac{1}{12}.\end{aligned}$$

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- Define P_c as the frequency of c-Ramanujan primes amongst the primes,
- As $N \to \infty$, $P_c = 1 - c$,
- For finite intervals $[a, b]$, P_c is a function of a and b,
- Choose $a = 10^5$, $b = 10^6$.

Distribution of Ramanujan primes (Sondow, Nicholson, Noe 2011)

	Length of the longest run in $[10^5, 10^6]$ of			
	c-Ramanujan primes		Non-c-Ramanujan primes	
c	Expected	Actual	Expected	Actual
0.50	14	20	16	36

Distribution of generalized c-Ramanujan primes (ABMRS 2011)

	Length of the longest run in $[10^5, 10^6]$ of			
	c-Ramanujan primes		Non-c-Ramanujan primes	
c	Expected	Actual	Expected	Actual
0.10	70	58	5	3
0.20	38	36	7	7
0.30	25	25	10	12
0.40	18	21	13	16
0.50	14	20	16	36
0.60	11	17	22	42
0.70	9	14	30	78
0.80	7	9	46	154
0.90	5	11	91	345

Open Problems

1. Sondow and Laishram: $p_{2n} < R_n < p_{3n}$ for $n > 1$. Can we find good choices of a_c and b_c such that $p_{a_c n} \leq R_{c,n} \leq p_{b_c n}$ for all n?

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2. For a given prime p, for what values of c is p a c-Ramanujan prime?
3. Is there any explanation for the unexpected distribution of c-Ramanujan primes amongst the primes?

Acknowledgments

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