The Distribution of Generalized Ramanujan Primes

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Combinatorial and Additive Number Theory (CANT 2012)
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Prime Numbers

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- \( \pi(x) \sim \frac{x}{\log x} \) (Prime Number Theorem).

- For fixed \( c \in (0, 1) \), we expect the amount of primes in an interval \((cx, x]\) to increase with \(x\).
Bertrand’s Postulate (1845)

For all integers \( x \geq 2 \), there exists at least one prime in \((x/2, x]\).
Ramanujan Primes

**Definition**

The $n$-th Ramanujan prime $R_n$ is the smallest integer such that for any $x \geq R_n$, at least $n$ primes are in $(x/2, x]$. 
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- Sondow: As $x \to \infty$, 50% of primes $\leq x$ are Ramanujan.
Definition
For $c \in (0, 1)$, the $n$-th $c$-Ramanujan prime $R_{c,n}$ is the smallest integer such that for any $x \geq R_{c,n}$, at least $n$ primes are in $(cx, x]$. 
Preliminaries

Let $\pi(x)$ be the prime-counting function that gives the number of primes less than or equal to $x$.

The Prime Number Theorem states:

$$
\lim_{x \to \infty} \frac{\pi(x)}{x / \log(x)} = 1.
$$
The logarithmic integral function $\text{Li}(x)$ is defined by

$$\text{Li}(x) = \int_2^x \frac{1}{\log t} dt.$$ 

The Prime Number Theorem gives us

$$\pi(x) = \text{Li}(x) + O\left(\frac{x}{\log^2 x}\right),$$

i.e., there is a $C > 0$ such that for all $x$ sufficiently large

$$-C \frac{x}{\log^2 x} \leq \pi(x) - \text{Li}(x) \leq C \frac{x}{\log^2 x}.$$
**Existence of $R_{c,n}$**

**Theorem (ABMRS 2011)**

For all $n \in \mathbb{Z}$ and all $c \in (0, 1)$, the $n$-th $c$-Ramanujan prime $R_{c,n}$ exists.
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- The number of primes in $(cx, x]$ is $\pi(x) - \pi(cx)$.
- Using the Prime Number Theorem and Mean Value Theorem, there exists a $b_c \in [0, -\log c]$, $\pi(x) - \pi(cx) = \frac{(1 - c)x}{\log x - b_c} + O \left( \frac{x}{\log^2 x} \right)$. 
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- Using the Prime Number Theorem and Mean Value Theorem, there exists a $b_c \in [0, -\log c]$, $\pi(x) - \pi(cx) \approx \frac{(1-c)x}{\log x - b_c} + O\left(\frac{x}{\log^2 x}\right)$.

- For any integer $n$ and for all $x$ sufficiently large, $\pi(x) - \pi(cx) \geq n$. 
Asymptotic Behavior

**Theorem (ABMRS 2011)**

For any fixed $c \in (0, 1)$, the $n$-th $c$-Ramanujan prime is asymptotic to the $\frac{n}{1-c}$-th prime as $n \to \infty$.

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**Sketch:**

By the triangle inequality

$$\left| R_{c,n} - p_{\frac{n}{1-c}} \right| \leq \left| R_{c,n} - \frac{n}{1-c} \log R_{c,n} \right| + \frac{n}{1-c} \log R_{c,n} - \frac{n}{1-c} \log n$$

$$+ \left| \frac{n}{1-c} \log n - \frac{n}{1-c} \log \frac{n}{1-c} \right|$$

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\leq \gamma_c n \log \log n.
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$$\leq \gamma_c n \log \log n.$$ 

Since $\frac{n \log \log n}{p_{\frac{n}{1-c}}} \to 0$ as $n \to \infty \Rightarrow R_{c,n} \sim p_{\frac{n}{1-c}}$. 

Asymptotic Behavior
Frequency of $c$-Ramanujan Primes

Theorem (ABMRS 2011)

In the limit, the probability of a generic prime being a $c$-Ramanujan prime is $1 - c$.

Sketch:

- Define $N = \left\lfloor \frac{n}{1-c} \right\rfloor$. 
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  ![Number Line](image)

  - $a_N$, $p_N$, $b_N$

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  - $R_{c,n} = a_N$ and every prime in $(a_N, p_N]$ is $c$-Ramanujan,
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- $R_{c,n} = b_N$ and every prime in $[p_N, b_N)$ is $c$-Ramanujan.

Goal: $\frac{\pi(b_N) - \pi(a_N)}{\pi(p_N)} \to 0$ as $N \to \infty$. 
Let:

\[ a_N = p_N - \beta_c N \log \log N, \quad b_N = p_N + \beta_c N \log \log N. \]
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$$a_N = p_N - \beta_c N \log \log N, \quad b_N = p_N + \beta_c N \log \log N.$$ 

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- Then, $R_{c,n} \in [a_N, b_N]$.

- Using the Prime Number Theorem, we can show

$$\frac{\pi(b_N) - \pi(a_N)}{\pi(p_N)} \leq \xi_c \frac{\log \log N}{\log N} \to 0 \text{ as } N \to \infty.$$
## Prime Numbers

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### Ramanujan Primes ($c = \frac{1}{2}$)

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Coin Flipping Model (Variation on Cramer Model)

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$$\mathbb{E}[L_N] \approx \frac{\log N}{\log(1/P)} - \left( \frac{1}{2} - \frac{\log(1 - P) + \gamma}{\log(1/P)} \right),$$
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$$

$$
\text{Var}[L_N] \approx \frac{\pi^2}{6 \log^2 (1/P)} + \frac{1}{12}.
$$
Define $P_c$ as the frequency of $c$-Ramanujan primes amongst the primes,
What is $P$?

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- Choose $a = 10^5$, $b = 10^6$. 
Distribution of Ramanujan primes (Sondow, Nicholson, Noe 2011)

<table>
<thead>
<tr>
<th>$c$</th>
<th>Length of the longest run in $[10^5, 10^6]$ of $c$-Ramanujan primes</th>
<th>Non-$c$-Ramanujan primes</th>
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<tbody>
<tr>
<td>0.50</td>
<td>$14$</td>
<td>$20$</td>
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### Distribution of generalized $c$-Ramanujan primes (ABMRS 2011)

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<tr>
<td></td>
<td>Expected</td>
<td>Actual</td>
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<td>0.10</td>
<td>70</td>
<td>58</td>
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<td>0.20</td>
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Open Problems

1. Sondow and Laishram: $p_{2n} < R_n < p_{3n}$ for $n > 1$. Can we find good choices of $a_c$ and $b_c$ such that $p_{ac}n \leq R_{c,n} \leq p_{bc}n$ for all $n$?
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2. For a given prime $p$, for what values of $c$ is $p$ a $c$-Ramanujan prime?
Open Problems

1. Sondow and Laishram: $p_{2n} < R_n < p_{3n}$ for $n > 1$. Can we find good choices of $a_c$ and $b_c$ such that $p_{a_cn} \leq R_{c,n} \leq p_{b_cn}$ for all $n$?

2. For a given prime $p$, for what values of $c$ is $p$ a $c$-Ramanujan prime?

3. Is there any explanation for the unexpected distribution of $c$-Ramanujan primes amongst the primes?
This work was supported by the NSF, Williams College, University College London.

We would like to thank our advisors Steven J. Miller and Jonathan Sondow, as well as our colleagues from the 2011 REU at Williams College.