Lowest zeros of GL(2) L-functions & arithmetic matrix ensembles

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Outline

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Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \varepsilon$ of the average spacing to the next zero.
- Granville, Soundararajan: Use 'weak' GRH for $L(s,\chi)$ (χ primitive mod q.) to obtain $\sum_{n \leqslant x} \chi(n) = o(x)$ whenever $x \geqslant q^{\varepsilon}$. (Burgess: $x \geqslant q^{1/4+\varepsilon}$.)

Our goals

$$\left\{\text{Tune RMT models to be sensitive to fine arithmetic.}\right\} \leftrightsquigarrow \left\{\text{Use insight from RMT to make conjectures about } L\text{-functions.}\right\}$$

Our setup

- Let $f \in S_k^\star(M, \chi_f)$ be a primitive holomorphic cusp form on $\Gamma_0(M)\backslash\mathfrak{H}$ of weight k with nebentypus χ_f that is an eigenfunction of all the Hecke operators.

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$$f(z) = \sum_{n>0} \lambda_f(n) n^{(k-1)/2} e(nz).$$

- The Dirichlet series admits an Euler product

$$\sum_{n\geqslant 1} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \lambda_f(p)p^{-s} + \chi_f(p)p^{-2s})^{-1}$$
$$= \prod_p (1 - \alpha_f(p)p^{-s})^{-1}(1 - \beta(p)p^{-s})^{-1};$$

both absolutely convergent for $\Re s > 1$ and may be continued to automorphic L-function on $\mathfrak{H}$ (fun. eq., GRH, Ramanujan…).

Our setup (cont'd)

For $d > 0$ a fundamental discriminant prime to M, let $\psi_d = \left(\frac{d}{\cdot}\right)$ be the Kronecker symbol. We may then twist $f \in S_k^\star(M, \chi_f)$ by ψ_d, writing

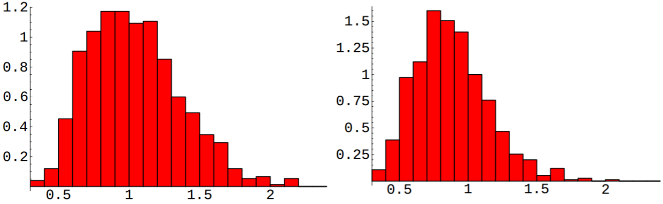
$$f \otimes \psi_d = \sum_{n>0} \lambda_f(n)\psi_d(n)n^{(k-1)/2}e(nz).$$

Then $f \otimes \psi_d =: f_d \in S_k^\star(Md^2, \chi_f\psi_d^2)$ with Fourier coefficients $\lambda_{f_d} = \lambda_f\psi_d$.

Outline

Repulsion in families of elliptic curve L-functions

- Significant discrepancy, '06: S. J. Miller observed that the lowest-lying zeros of elliptic curve L-functions of finite conductor were repulsed from the central point.

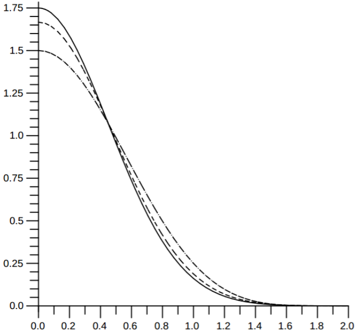


1[st] normalized zero above central point of rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ (from DHKMS).

L: 750 curves with log(cond) $\in [3.2, 12.6]$. R: 750 curves with log(cond) $\in [12.6, 14.9]$.

Repulsion in families of elliptic curve L-functions

Not predicted by random matrix theory $\rightsquigarrow$ missing arithmetic ingredient?



Probability density of normalized eigenvalue closest to 1 for SO(8) (solid), SO(6) (dashed) and SO(4) (dot-dashed). (From DHKMS.)

Repulsion in families of elliptic curve L-functions

- Formulæ of Waldspurger $\rightsquigarrow$ Kohnen-Zagier relate the central value of the twisted L-function $L(s, f \otimes \psi_d)$ to the Fourier coefficient $c(|d|)$ of a half-integral weight modular form associated to f by the Shimura correspondance.

$$L(1/2, f \otimes \psi_d) \;=\; \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}}.$$

- Basic observation: $L(1/2, f \otimes \psi_d)$ *discretized at the central point* $\Rightarrow$ cutoff

$$L(1/2, f \otimes \psi_d) < \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}} \Rightarrow L(1/2, f \otimes \psi_d) = 0.$$

No KZ with nebentypus

N.B.: there is no Kohnen-Zagier-style formula for forms with nebentypus.

- Is there repulsion?

Symmetry types

- For the family of ‘even’ quadratic twists of a form without nebentypus, symmetry type is SO(even).
- What about a generic f with nontrivial nebentypus? f with complex multiplication by its own nebentypus?

Convergence to the RMT limit

What's the right matrix size?

- RMT + Katz-Sarnak describe the *limiting behavior* for large random matrices of dimension N as $N \to \infty$. This should also describe the limit in the asymptotic parameters of appropriate families of L-functions.
- How well do the classical matrix groups model local statistics of L-functions *outside* the scaling limit?
  - How we approach the limit of large N comes into play.

Convergence to the RMT limit

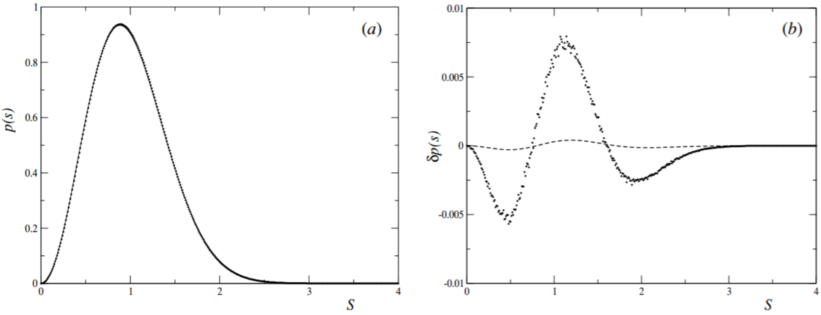
Case of $\zeta(s)$

- Bogomolny, Bohigas, Leboeuf & Monastra compare the difference between the asymptotic and finite N_0 nearest-neighbor spacing for CUE matrices to that for ζ zeros, where

$$N_0 = \log \frac{E}{2\pi}$$

is obtained by equating the local density of zeros of $\zeta(s)$ with the local density of eigenvalues of matrices in $U(N)$.

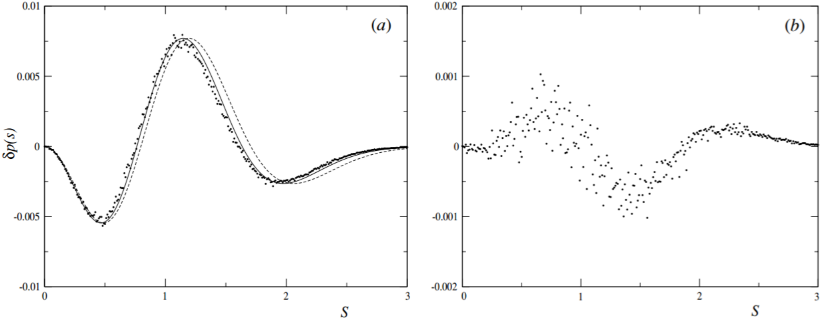
Convergence to the RMT limit: Bogomolny *et al.*



L: Odlyzko's 70 million $\zeta(s)$ nearest-neighbor spacings.

R: Difference between numerical result for $\zeta(s)$ and asymptotic CUE curve (dots) compared to difference between spacing distribution of CUE of size N_0 and asymptotic curve (dashed line). (From B. *et al.*.)

Convergence to the RMT limit: Bogomolny *et al.*



L: Difference between nearest-neighbor spacing of Riemann zeros and the asymptotic CUE for a billion zeros in a window near 2.504×10^{15} (dots) compared to theory that takes into acccount arithmetic of lower order terms (full line). (From B. *et al.*.)

R: Difference between ζ zeros (dots) and theory (full line) ($\rightsquigarrow O(N^{-4})$ correction?).

Outline

An arithmetic matrix ensemble: the EOM

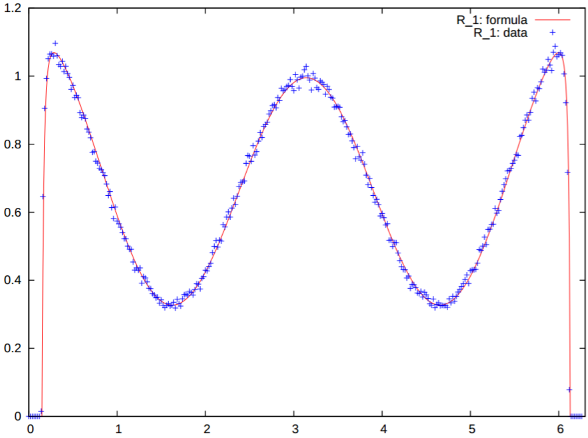
- Dueñez, Huynh, Keating, Miller '11: new random matrix model for elliptic curve L-functions of finite conductor.
- 2 main (arithmetic) ingredients:
  - Cutoff value to model discretization of L-values at the central point.
  - Effective matrix size as introduced by Bogomolny *et al.* to better model $\zeta(s)$.

Key properties of EOM one-level density

After showing that the one-level density $R_1^{T\mathcal{X}}(\theta)$ of the EOM can be 'completely understood' in terms of ratios of gamma factors and elementary functions, DHKMS prove that

$$R_1^{T\mathcal{X}}(\theta) = \begin{cases} 0 & \text{for } d(\theta, \mathcal{X}) < 0 \\ R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k(\theta) \exp((k+1/2)\mathcal{X}) & \text{for } d(\theta, \mathcal{X}) \geqslant 0, \end{cases}$$

for some coefficients b_k, where
$d(\theta, \mathcal{X}) = (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$.
We see that in fact $R_1^{T\mathcal{X}}$ exhibits a 'hard gap' near 0 and in the limit $\mathcal{X} \to -\infty$ with θ fixed, $R_1^{T\mathcal{X}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.



One-level density of matrices from SO$(2N)$ with $N = 2$, cutoff $|\Lambda_A(1,N)| \geqslant 0.1$, from DHKMS. Red curve: formula for $R_1^{T_\mathcal{X}}(\theta)$; blue crosses: 200,000 numerically generated matrices. (From DHKMS.)

Outline

EOM cutoff value

Restrict now to $f \in S_k^\star(M)$. Notion of even twists well-defined.
Define a pair of moment generating functions

$$M_f(X,s) := \frac{1}{\left|\mathscr{D}_f(X)\right|} \sum_{d \in \mathscr{D}_f(X)} L_f(1/2, \psi_d)^s, \quad \text{and}$$

$$M_{\mathrm{SO}(2N)}(N,s) := \int_{\mathrm{SO}(2N)} \Lambda_A(1,N)^s \, \mathrm{d}A.$$

Expect

$$M_f(X,s) \sim a_f(s) M_{W(f)}(N,s),$$

then relate probability density functions $P_f(d,x)$ and $P_{O^+}(d,x)$.

Guessing the order of vanishing

If we let Y_d be a random variable with probability density $P_f(d,x)$, we might suggest that (following Conrey, Keating, Rubinstein & Snaith), for our family $\mathscr{F}^+(X)$ of even twists,

$$\left|\left\{L_f(s,\psi_d) \in \mathscr{F}^+(X) : d \text{ prime}, L_f(1/2,\psi_d) = 0\right\}\right| \\ = \sum_{\substack{d \leqslant X \\ d \text{ prime}}}^{\star} \text{Prob}\left(0 \leqslant Y_d \leqslant \frac{\delta_f \kappa_f}{d^{(k-1)/2}}\right),$$

up to some constant δ_f. (Starred sum restricts to even d.)

δ_f

$$\left|\left\{L_f(s,\psi_d) \in \mathscr{F}^+(X) : d \text{ prime}, L_f(1/2,\psi_d) = 0\right\}\right| = \sum_{\substack{d\leqslant X \\ d \text{ prime}}}^{\star} \text{Prob}\left(0 \leqslant Y_d \leqslant \frac{\delta_f \kappa_f}{d^{(k-1)/2}}\right).$$

What is δ_f? Hard to know, so we determine numerically.

Easy to translate cutoff into scaled cutoff for distribution of values of characteristic polynomial.

Ratios calculation

- Main innovation in Bogomolny *et al.* is to introduce an effective matrix size to better capture the arithmetic of lower-order terms.
- Want to match lower-order terms for local statistics on RMT side to lower-order terms on arithmetic/L side.

Ratios calculation

- Start with a ratio of L-functions

$$\sum_{\substack{0<d\leqslant X \\ d \text{ good}}} \frac{L\left(f\otimes\psi_d, \frac{1}{2}+\alpha\right)}{L\left(f\otimes\psi_d, \frac{1}{2}+\gamma\right)}.$$

- Apply approximate functional equation to the numerator while replacing the denominator by its reciprocal Dirichlet series.
- Complete resulting sums, discard remainder from the approximate functional equation, and retain only the diagonal, replacing oscillatory terms with their expectation.
- Left with an Euler product. Factor out divergent factors.
- Conjecture cancellation up to $X^{1/2+\varepsilon}$.

One-level density

Corollary 1.

One-level density for the scaled zeros of the family $\mathscr{F}(X)$, a family of quadratic twists of the L-function of a holomorphic cuspidal newform f of weight k and (odd prime) level M, is

$$\frac{1}{\left|\mathscr{D}_f(X)\right|} S_1(f) = \int_{-\infty}^{\infty} g(\tau)\left(1 + \frac{\sin(2\pi\tau)}{2\pi\tau} + a_1 \frac{1+\cos(2\pi\tau)}{R} - a_2 \frac{\pi\tau\sin(2\pi\tau)}{R^2} + O\left(R^{-3}\right)\right) \mathrm{d}\tau$$

if χ_f is principal,

$$\frac{1}{\left|\mathscr{D}_f(X)\right|} S_1(f) = \int_{-\infty}^{\infty} g(\tau)\left(1 + \frac{b_1 + b_2\cos(2\pi\tau)}{R} + c_1 \frac{\pi\tau\sin(2\pi\tau)}{R^2} + O\left(R^{-3}\right)\right) \mathrm{d}\tau$$

if χ_f is not principal and $f \neq \overline{f}$, and

$$\frac{1}{\left|\mathscr{D}_f(X)\right|} S_1(f) = \int_{-\infty}^{\infty} g(\tau)\left(1 - \frac{\sin(2\pi\tau)}{2\pi\tau} + d_1 \frac{1-\cos(2\pi\tau)}{R} + d_2 \frac{\pi\tau\sin(2\pi\tau)}{R^2} + O\left(R^{-3}\right)\right) \mathrm{d}\tau,$$

if χ_f is not principal and $f = \overline{f}$.

One-level density

Sample arithmetic term:

$$a_1 = -1 - \gamma + \psi\left(\frac{k}{2}\right) + A_f^1(0,0) + \frac{L_f'(\mathrm{sym}^2,1)}{L_f(\mathrm{sym}^2,1)}$$

and

$$a_2 = 2 + 2\left(\gamma - \psi\left(\frac{k}{2}\right)\right) + \psi\left(\frac{k}{2}\right)^2 - 2\gamma\psi\left(\frac{k}{2}\right) - 2\gamma_1 + \left(1 + \gamma - \psi\left(\frac{k}{2}\right)\right) B_f'(0)$$
$$+ \frac{B_f''(0)}{4} - 2\left(1 + \gamma - \psi\left(\frac{k}{2}\right) + \frac{B_f'(0)}{2}\right) \frac{L_f'(\mathrm{sym}^2,1)}{L_f\left(\mathrm{sym}^2,1\right)} + \frac{L_f''(\mathrm{sym}^2,1)}{L_f\left(\mathrm{sym}^2,1\right)}$$

Match terms. But: no terms to match for $U(N)$. Pair-correlation.

(Scaled) Pair-correlation expansion

$$\sum_{0<\gamma,\gamma'<T} \varphi\left((\gamma-\gamma')\frac{R}{\pi}\right) = \frac{T}{\pi}\log\left(\frac{\sqrt{M}|d|T}{2\pi e}\right)\Bigg(g(0)+$$

$$+\int_{-\infty}^{\infty} g(y)\left(1-\left(\frac{\sin\pi y}{\pi y}\right)^2 - \frac{e_1\sin^2\pi y}{R^2} - \frac{e_2\pi y\sin 2\pi y}{R^3} + O\left(R^{-4}\right)\right)\mathrm{d}y\Bigg) + O\left(T^{1/2+\varepsilon}\right)$$

$$\sim N\left(T, f\otimes\psi_d\right)\left(g(0)+\int_{-\infty}^{\infty} g(y)\left(1-\left(\frac{\sin\pi y}{\pi y}\right)^2\right)\mathrm{d}y\right),$$

The (scaled) pair-correlation $\mathcal{Q}_{U(N)}(v)$ for $U(N)$ is

$$\mathcal{Q}_{U(N)}(v) = 1-\left(\frac{1}{N}\frac{\sin(\pi v)}{\sin(\pi v/N)}\right)^2 = 1-\left(\frac{\sin\pi v}{\pi v}\right)^2 - \frac{\sin^2\pi v}{3N^2} + O\left(N^{-4}\right).$$

For this ensemble, we select $N_{\text{eff}} = R/\sqrt{3e_1}$.

Outline

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- Steven J. Miller – *sjm1@williams.edu*
- Patrick Ryan – *patrickryan01@college.harvard.edu*

Thank You!

Outline

Ratios Conjecture

Conjecture (Ratios Conjecture).

For some reasonable conditions such as $-\frac{1}{4} < \Re(\alpha) < \frac{1}{4}$, $\frac{1}{\log X} \ll \Re(\gamma) < \frac{1}{4}$ *and* $\Im(\alpha), \Im(\gamma) \ll X^{1-\varepsilon}$*, we have*

$$R_f(\alpha,\gamma) = \sum_{d\in\mathscr{D}_f(X)} \frac{L_f\left(\frac{1}{2}+\alpha,\psi_d\right)}{L_f\left(\frac{1}{2}+\gamma,\psi_d\right)}$$

$$= \sum_{d\in\mathscr{D}_f(X)} \left[Y_f A_f(\alpha,\gamma) + \eta_f \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2\alpha} \frac{\Gamma\left(\frac{k}{2}-\alpha\right)}{\Gamma\left(\frac{k}{2}+\alpha\right)} \tilde{Y}_f \tilde{A}_f(-\alpha,\gamma)\right]$$

$$+ O\left(X^{1/2+\varepsilon}\right),$$

where A_f*, and* $\tilde{A}_f$ *are analytic Euler products near 1,* M *is the (odd prime) level of the L-function* $L_f(s)$*,* ϵ_f *is the root number of its functional equation,*

Ratios Conjectures

where

$$\tilde{Y}_f(-\alpha,\gamma) = \frac{L\left(\chi'_f, 1+2\gamma\right) L_{\overline{f}}(\mathrm{sym}^2, 1-2\alpha)}{L(f \otimes \overline{f}, 1-\alpha+\gamma)},$$

and

$$\tilde{Y}_f(-\alpha,\gamma) = \frac{L\left(\chi'_f, 1+2\gamma\right) L_{\overline{f}}(\mathrm{sym}^2, 1-2\alpha)}{L(f \otimes \overline{f}, 1-\alpha+\gamma)}.$$

One-level density

$$
\begin{aligned}
&S_1(f)\\
&= \sum_{d\in\mathscr{D}_f(X)} \sum_{\gamma_d} \varphi(\gamma_d)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) \sum_{d\in\mathscr{D}_f(X)} \Bigg[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}+it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}-it\right)\\
&\qquad - \frac{L'(\chi'_f, 1+2it)}{L(\chi'_f, 1+2it)} + \frac{L'_f(\mathrm{sym}^2, 1+2it)}{L_f(\mathrm{sym}^2, 1+2it)} + A^1_f(it,it)\\
&\qquad - \eta_f \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} \frac{\Gamma\left(\frac{k}{2}-it\right)}{\Gamma\left(\frac{k}{2}+it\right)} \frac{L\left(\chi'_f, 1+2it\right) L_{\overline{f}}(\mathrm{sym}^2, 1-2it)}{L_f(\mathrm{ad}^2, 1)} \tilde{A}_f(-it,it)\\
&\qquad - \frac{L'(\chi'_{\overline{f}}, 1+2it)}{L(\chi'_{\overline{f}}, 1+2it)} + \frac{L'_{\overline{f}}(\mathrm{sym}^2, 1+2it)}{L_{\overline{f}}(\mathrm{sym}^2, 1+2it)} + A^1_{\overline{f}}(it,it)\\
&\qquad - \eta_{\overline{f}} \left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2it} \frac{\Gamma\left(\frac{k}{2}-it\right)}{\Gamma\left(\frac{k}{2}+it\right)} \frac{L\left(\chi'_{\overline{f}}, 1+2it\right) L_f(\mathrm{sym}^2, 1-2it)}{L_{\overline{f}}(\mathrm{ad}^2, 1)} \tilde{A}_{\overline{f}}(-it,it) \Bigg] \,\mathrm{d}t\\
&\qquad + O(X^{1/2+\varepsilon})
\end{aligned}
$$

Pair-correlation

Theorem 2.

Assuming the Ratios Conjecture, and with φ satisfying some weak conditions, we have

$$P(\varphi) = \sum_{0<\gamma,\gamma'<T} \varphi\left(\gamma-\gamma'\right) = \frac{2}{(2\pi)^2}\int_0^T \Bigg[2\pi\varphi(0)\log\left(\frac{\sqrt{M}|d|t}{2\pi}\right) + \int_{-T}^{T}\varphi(r)\Bigg(2\log^2\left(\frac{\sqrt{M}|d|t}{2\pi}\right)$$

$$+\left(\frac{L'_\star}{L_\star}\right)'\left(1+ir,(f\otimes\psi_d)\otimes(\overline{f}\otimes\psi_d)\right)$$

$$+\frac{1}{c^2_{f\otimes\psi_d}}\left(\frac{\sqrt{M}|d|t}{2\pi}\right)^{-2ir} L(1+ir,(f\otimes\psi_d)\otimes(\overline{f}\otimes\psi_d))L(1-ir,(f\otimes\psi_d)\otimes(\overline{f}\otimes\psi_d))\mathscr{A}(ir)$$

$$-\mathscr{B}(1+ir)\Bigg)\,\mathrm{d}r\Bigg]\,\mathrm{d}t + O\left(T^{1/2+\varepsilon}\right),$$

where the inner integral is to be regarded as a principal value near $r=0$, f is new of level M, $M|d|^2 = N$, ψ_d is the Kronecker character associated to a fundamental discriminant $d>0$ prime to M, $c_{f\otimes\psi_d} = \mathrm{res}_{s=1} L(s,(f\otimes\psi_d)\otimes(\overline{f}\otimes\psi_d))$ is an arithmetic constant, $\mathscr{A}(\rho)$ and $\mathscr{B}(1+\rho)$ are Euler factors analytic near 1, and $L_\star(s,(f\otimes\psi)\otimes(\overline{f}\otimes\psi_d))$ is the 'unramified' part of the Rankin-Selberg convolution.