Lowest zeros of $GL(2)$ $L$-functions & arithmetic matrix ensembles

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Outline

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Motivating questions

An arithmetic matrix ensemble: the Excised Orthogonal Model

New EOM for cusp forms
  Cutoff
    Effective matrix size & symmetry types

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Why study zeros of $L$-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev’s bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ ‘most’ of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from $L$-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \varepsilon$ of the average spacing to the next zero.
- Granville, Soundararajan: Use ‘weak’ GRH for $L(s, \chi)$ ($\chi$ primitive mod $q$.) to obtain $\sum_{n \leq x} \chi(n) = o(x)$ whenever $x \geq q^{\varepsilon}$. (Burgess: $x \geq q^{1/4+\varepsilon}$.)
Our goals

\[
\begin{align*}
\{ \text{Tune RMT models to be sensitive to fine arithmetic.} \} & \iff \{ \text{Use insight from RMT to make conjectures about } L\text{-functions.} \}
\end{align*}
\]
Our setup

- Let $f \in S_k^*(M, \chi_f)$ be a primitive holomorphic cusp form on $\Gamma_0(M) \backslash \mathcal{H}$ of weight $k$ with nebentypus $\chi_f$ that is an eigenfunction of all the Hecke operators.
Our setup

- Let \( f \in S^*_k(M, \chi_f) \) be a primitive holomorphic cusp form on \( \Gamma_0(M) \backslash \mathcal{H} \) of weight \( k \) with nebentypus \( \chi_f \) that is an eigenfunction of all the Hecke operators.
- \( f \) has a \( q \)-expansion at \( \infty \) given by

\[
f(z) = \sum_{n>0} \lambda_f(n) n^{(k-1)/2} e(nz).
\]
Our setup

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- $f$ has a $q$-expansion at $\infty$ given by

$$f(z) = \sum_{n>0} \lambda_f(n) n^{(k-1)/2} e(nz).$$

- The Dirichlet series admits an Euler product

$$\sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \frac{1}{(1 - \lambda_f(p) p^{-s} + \chi_f(p) p^{-2s})^{-1}}$$

$$= \prod_p \frac{1}{(1 - \alpha_f(p) p^{-s})^{-1}(1 - \beta(p) p^{-s})^{-1}};$$

both absolutely convergent for $\Re s > 1$ and may be continued to automorphic $L$-function on $\mathcal{H}$ (fun. eq., GRH, Ramanujan...).
Our setup (cont’d)

For $d > 0$ a fundamental discriminant prime to $M$, let $\psi_d = \left(\frac{d}{.}\right)$ be the Kronecker symbol. We may then twist $f \in S_k^*(M, \chi_f)$ by $\psi_d$, writing

$$f \otimes \psi_d = \sum_{n > 0} \lambda_f(n) \psi_d(n) n^{(k-1)/2} e(nz).$$

Then $f \otimes \psi_d =: f_d \in S_k^*(Md^2, \chi_f \psi_d^2)$ with Fourier coefficients $\lambda_{f_d} = \lambda_f \psi_d$. 
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Repulsion in families of elliptic curve $L$-functions

- **Significant discrepancy, ’06:** S. J. Miller observed that the lowest-lying zeros of elliptic curve $L$-functions of finite conductor were repulsed from the central point.

1\textsuperscript{st} normalized zero above central point of rank 0 curves from $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ (from DHKMS).

\textbf{L:} 750 curves with log(cond) $\in [3.2, 12.6]$. \textbf{R:} 750 curves with log(cond) $\in [12.6, 14.9]$. 
Repulsion in families of elliptic curve $L$-functions

Not predicted by random matrix theory \( \rightsquigarrow \) missing arithmetic ingredient?

![Probability density of normalized eigenvalue closest to 1 for SO(8) (solid), SO(6) (dashed) and SO(4) (dot-dashed). (From DHKMS.)](image-url)
Repulsion in families of elliptic curve $L$-functions

- Formulæ of Waldspurger $\mapsto$ Kohnen-Zagier relate the central value of the twisted $L$-function $L(s, f \otimes \psi_d)$ to the Fourier coefficient $c(|d|)$ of a half-integral weight modular form associated to $f$ by the Shimura correspondance.

\[
L(1/2, f \otimes \psi_d) = \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}}.
\]

- **Basic observation:** $L(1/2, f \otimes \psi_d)$ discretized at the central point $\Rightarrow$ cutoff

\[
L(1/2, f \otimes \psi_d) < \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}} \Rightarrow L(1/2, f \otimes \psi_d) = 0.
\]
No KZ with nebentypus

N.B.: there is no Kohnen-Zagier-style formula for forms with nebentypus.

- Is there repulsion?
Symmetry types

- For the family of ‘even’ quadratic twists of a form without nebentypus, symmetry type is SO(even).

- What about a generic $f$ with nontrivial nebentypus? $f$ with complex multiplication by its own nebentypus?
Convergence to the RMT limit

What’s the right matrix size?

- RMT + Katz-Sarnak describe the \textit{limiting behavior} for large random matrices of dimension $N$ as $N \to \infty$. This should also describe the limit in the asymptotic parameters of appropriate families of $L$-functions.

- \textbf{How well do the classical matrix groups model local statistics of $L$-functions \textit{outside} the scaling limit?}
  - How we approach the limit of large $N$ comes into play.
Convergence to the RMT limit

Case of $\zeta(s)$

- Bogomolny, Bohigas, Leboeuf & Monastra compare the difference between the asymptotic and finite $N_0$ nearest-neighbor spacing for CUE matrices to that for $\zeta$ zeros, where

$$N_0 = \log \frac{E}{2\pi}$$

is obtained by equating the local density of zeros of $\zeta(s)$ with the local density of eigenvalues of matrices in $U(N)$.
Convergence to the RMT limit: Bogomolny *et al.*

**L:** Odlyzko’s 70 million \( \zeta(s) \) nearest-neighbor spacings.

**R:** Difference between numerical result for \( \zeta(s) \) and asymptotic CUE curve (dots) compared to difference between spacing distribution of CUE of size \( N_0 \) and asymptotic curve (dashed line). (From B. *et al.*.)
Convergence to the RMT limit: Bogomolny et al.

L: Difference between nearest-neighbor spacing of Riemann zeros and the asymptotic CUE for a billion zeros in a window near $2.504 \times 10^{15}$ (dots) compared to theory that takes into account arithmetic of lower order terms (full line). (From B. et al.)

R: Difference between $\zeta$ zeros (dots) and theory (full line) ($\sim O(N^{-4})$ correction?).
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An arithmetic matrix ensemble: the EOM

- Dueñez, Huynh, Keating, Miller ’11: new random matrix model for elliptic curve $L$-functions of finite conductor.
- 2 main (arithmetic) ingredients:
  - **Cutoff value** to model discretization of $L$-values at the central point.
  - **Effective matrix size** as introduced by Bogomolny *et al.* to better model $\zeta(s)$. 
Key properties of EOM one-level density

After showing that the one-level density $R_1^{T\chi}(\theta)$ of the EOM can be ‘completely understood’ in terms of ratios of gamma factors and elementary functions, DHKMS prove that

$$R_1^{T\chi}(\theta) = \begin{cases} 0 & \text{for } d(\theta, \chi) < 0 \\ R_1^{SO(2N)}(\theta) + C\chi \sum_{k=0}^{\infty} b_k(\theta) \exp((k + 1/2)\chi) & \text{for } d(\theta, \chi) \geq 0, \end{cases}$$

for some coefficients $b_k$, where

$$d(\theta, \chi) = (2N - 1) \log 2 + \log(1 - \cos \theta) - \chi.$$

We see that in fact $R_1^{T\chi}$ exhibits a ‘hard gap’ near 0 and in the limit $\chi \to -\infty$ with $\theta$ fixed, $R_1^{T\chi}(\theta) \to R_1^{SO(2N)}(\theta)$. 


One-level density of matrices from \( \text{SO}(2N) \) with \( N = 2 \), cutoff \( |\Lambda_A(1, N)| \geq 0.1 \), from DHKMS. Red curve: formula for \( R_1^{\Sigma \chi} (\theta) \); blue crosses: 200,000 numerically generated matrices. (From DHKMS.)
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EOM cutoff value

Restrict now to $f \in S_k^\star(M)$. Notion of even twists well-defined. Define a pair of moment generating functions

$$M_f(X, s) := \frac{1}{|D_f(X)|} \sum_{d \in D_f(X)} L_f(1/2, \psi_d)^s, \quad \text{and}$$

$$M_{SO(2N)}(N, s) := \int_{SO(2N)} \Lambda_A(1, N)^s \, dA.$$  

Expect

$$M_f(X, s) \sim a_f(s) M_{W(f)}(N, s),$$

then relate probability density functions $P_f(d, x)$ and $P_{O^+}(d, x)$. 
Guessing the order of vanishing

If we let $Y_d$ be a random variable with probability density $P_f(d, x)$, we might suggest that (following Conrey, Keating, Rubinstein & Snaith), for our family $\mathcal{F}^+(X)$ of even twists,

$$\left|\{L_f(s, \psi_d) \in \mathcal{F}^+(X) : d \text{ prime}, L_f(1/2, \psi_d) = 0\}\right|$$

$$= \sum_{{d \leq X \atop d \text{ prime}}} \star \text{Prob} \left( 0 \leq Y_d \leq \frac{\delta_f \kappa_f}{d^{(k-1)/2}} \right),$$

up to some constant $\delta_f$. (Starred sum restricts to even $d$.)
What is $\delta_f$? Hard to know, so we determine numerically.

Easy to translate cutoff into scaled cutoff for distribution of values of characteristic polynomial.
Ratios calculation

- Main innovation in Bogomolny *et al.* is to introduce an **effective matrix size** to better capture the arithmetic of lower-order terms.

- Want to match lower-order terms for local statistics on RMT side to lower-order terms on arithmetic/$L$ side.
Ratios calculation

- Start with a ratio of $L$-functions

\[ \sum_{0<d\leq X \text{ good}} \frac{L(f \otimes \psi_d, \frac{1}{2} + \alpha)}{L(f \otimes \psi_d, \frac{1}{2} + \gamma)}. \]

- Apply approximate functional equation to the numerator while replacing the denominator by its reciprocal Dirichlet series.

- Complete resulting sums, discard remainder from the approximate functional equation, and retain only the diagonal, replacing oscillatory terms with their expectation.

- Left with an Euler product. Factor out divergent factors.

- Conjecture cancellation up to $X^{1/2+\varepsilon}$.
One-level density

Corollary 1.

One-level density for the scaled zeros of the family $\mathcal{F}(X)$, a family of quadratic twists of the $L$-function of a holomorphic cuspidal newform $f$ of weight $k$ and (odd prime) level $M$, is

$$\frac{1}{|D_f(X)|} S_1(f) = \int_{-\infty}^{\infty} g(\tau) \left( 1 + \frac{\sin(2\pi \tau)}{2\pi \tau} + a_1 \frac{1 + \cos(2\pi \tau)}{R} - a_2 \frac{\pi \tau \sin(2\pi \tau)}{R^2} + O\left(R^{-3}\right) \right) d\tau$$

if $\chi_f$ is principal,

$$\frac{1}{|D_f(X)|} S_1(f) = \int_{-\infty}^{\infty} g(\tau) \left( 1 + \frac{b_1 + b_2 \cos(2\pi \tau)}{R} + c_1 \frac{\pi \tau \sin(2\pi \tau)}{R^2} + O\left(R^{-3}\right) \right) d\tau$$

if $\chi_f$ is not principal and $f \neq \bar{f}$, and

$$\frac{1}{|D_f(X)|} S_1(f) = \int_{-\infty}^{\infty} g(\tau) \left( 1 - \frac{\sin(2\pi \tau)}{2\pi \tau} + d_1 \frac{1 - \cos(2\pi \tau)}{R} + d_2 \frac{\pi \tau \sin(2\pi \tau)}{R^2} + O\left(R^{-3}\right) \right) d\tau,$$

if $\chi_f$ is not principal and $f = \bar{f}$. 
Effective matrix size & symmetry types

One-level density

Sample arithmetic term:

\[ a_1 = -1 - \gamma + \psi \left( \frac{k}{2} \right) + A_f^1(0, 0) + \frac{L'_f(\text{sym}^2, 1)}{L_f(\text{sym}^2, 1)} \]

and

\[
\begin{align*}
  a_2 &= 2 + 2 \left( \gamma - \psi \left( \frac{k}{2} \right) \right) + \psi \left( \frac{k}{2} \right)^2 - 2\gamma\psi \left( \frac{k}{2} \right) - 2\gamma_1 + \left( 1 + \gamma - \psi \left( \frac{k}{2} \right) \right) B'_f(0) \\
  &\quad + \frac{B''_f(0)}{4} - 2 \left( 1 + \gamma - \psi \left( \frac{k}{2} \right) + \frac{B'_f(0)}{2} \right) \frac{L'_f(\text{sym}^2, 1)}{L_f(\text{sym}^2, 1)} + \frac{L''_f(\text{sym}^2, 1)}{L_f(\text{sym}^2, 1)}
\end{align*}
\]

Match terms. But: no terms to match for \( U(N) \). Pair-correlation.
Effective matrix size & symmetry types

**(Scaled) Pair-correlation expansion**

\[
\sum_{0 < \gamma, \gamma' < T} \varphi \left( \frac{(\gamma - \gamma') R}{\pi} \right) = \frac{T}{\pi} \log \left( \frac{\sqrt{M|d|T}}{2\pi e} \right) \left( g(0) + \int_{-\infty}^{\infty} g(y) \left( 1 - \left( \frac{\sin \pi y}{\pi y} \right)^2 - \frac{e_1 \sin^2 \pi y}{R^2} - \frac{e_2 \pi y \sin 2\pi y}{R^3} + O\left( R^{-4} \right) \right) dy \right) + O\left( T^{1/2+\varepsilon} \right)
\]

\[
\sim N(T, f \otimes \psi_d) \left( g(0) + \int_{-\infty}^{\infty} g(y) \left( 1 - \left( \frac{\sin \pi y}{\pi y} \right)^2 \right) dy \right),
\]

The (scaled) pair-correlation \( Q_{U(N)}(v) \) for \( U(N) \) is

\[
Q_{U(N)}(v) = 1 - \left( \frac{\sin(\pi v)}{N \sin(\pi v / N)} \right)^2 = 1 - \left( \frac{\sin \pi v}{\pi v} \right)^2 - \frac{\sin^2 \pi v}{3N^2} + O\left( N^{-4} \right).
\]

For this ensemble, we select \( N_{\text{eff}} = R / \sqrt{3e_1} \).
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Ratios Conjecture

Conjecture (Ratios Conjecture).

For some reasonable conditions such as $-\frac{1}{4} < \Re(\alpha) < \frac{1}{4}$, $\frac{1}{\log X} \ll \Re(\gamma) < \frac{1}{4}$, and $\Im(\alpha), \Im(\gamma) \ll X^{1-\varepsilon}$, we have

$$R_f(\alpha, \gamma) = \sum_{d \in \mathcal{D}_f(X)} \frac{L_f \left( \frac{1}{2} + \alpha, \psi_d \right)}{L_f \left( \frac{1}{2} + \gamma, \psi_d \right)}$$

$$= \sum_{d \in \mathcal{D}_f(X)} \left[ Y_f A_f(\alpha, \gamma) + \eta_f \left( \frac{\sqrt{M|d|}}{2\pi} \right)^{-2\alpha} \frac{\Gamma \left( \frac{k}{2} - \alpha \right)}{\Gamma \left( \frac{k}{2} + \alpha \right)} \tilde{Y}_f \tilde{A}_f(-\alpha, \gamma) \right]$$

$$+ O \left( X^{1/2+\varepsilon} \right),$$

where $A_f$, and $\tilde{A}_f$ are analytic Euler products near 1, $M$ is the (odd prime) level of the $L$-function $L_f(s)$, $\epsilon_f$ is the root number of its functional equation,
Ratios Conjectures

where

\[ \tilde{Y}_f(-\alpha, \gamma) = \frac{L\left(\chi'_f, 1 + 2\gamma\right) L_f^{(\text{sym}^2, 1 - 2\alpha)}}{L(f \otimes f, 1 - \alpha + \gamma)}, \]

and

\[ \tilde{Y}_f(-\alpha, \gamma) = \frac{L\left(\chi'_f, 1 + 2\gamma\right) L_f^{(\text{sym}^2, 1 - 2\alpha)}}{L(f \otimes f, 1 - \alpha + \gamma)}. \]
One-level density

\[ S_1(f) = \sum_{d \in \mathcal{D}_f(X)} \sum_{\gamma_d} \varphi(\gamma_d) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) \sum_{d \in \mathcal{D}_f(X)} \left[ 2\log \left( \frac{\sqrt{M|d|}}{2\pi} \right) + \frac{\Gamma'(k/2 + it)}{\Gamma(k/2 + it)} + \frac{\Gamma'(k/2 - it)}{\Gamma(k/2 - it)} \right. \]

\[ - \frac{L'(\chi'_f, 1 + 2it)}{L(\chi'_f, 1 + 2it)} - \frac{L'(\text{sym}^2, 1 + 2it)}{L_f(\text{sym}^2, 1 + 2it)} + A_1^f(it, it) \]

\[ - \eta_f \left( \frac{\sqrt{M|d|}}{2\pi} \right)^{-2it} \left( \frac{\Gamma(k/2 - it)}{\Gamma(k/2 + it)} \right) \frac{L(\chi'_f, 1 + 2it) L_f(\text{sym}^2, 1 - 2it)}{L_f(\text{sym}^2, 1 + 2it)} \tilde{A}_f(-it, it) \]

\[ - \frac{L'(\chi'_{\bar{f}}, 1 + 2it)}{L(\chi'_{\bar{f}}, 1 + 2it)} + \frac{L'_f(\text{sym}^2, 1 + 2it)}{L_f(\text{sym}^2, 1 + 2it)} + A_1^f(it, it) \]

\[ - \eta_{\bar{f}} \left( \frac{\sqrt{M|d|}}{2\pi} \right)^{-2it} \left( \frac{\Gamma(k/2 - it)}{\Gamma(k/2 + it)} \right) \frac{L(\chi'_{\bar{f}}, 1 + 2it) L_f(\text{sym}^2, 1 - 2it)}{L_f(\text{sym}^2, 1 + 2it)} \tilde{A}_{\bar{f}}(-it, it) \]

\[ \int dt + O(X^{3/2 + \varepsilon}) \]
Pair-correlation

Theorem 2.

Assuming the Ratios Conjecture, and with $\varphi$ satisfying some weak conditions, we have

$$P(\varphi) = \sum_{0 < \gamma, \gamma' < T} \varphi(\gamma - \gamma') = \frac{2}{(2\pi)^2} \int_0^T 2\pi \varphi(0) \log \left( \frac{\sqrt{M|d|t}}{2\pi} \right) + \int_{-T}^T \varphi(r) \left( 2\log^2 \left( \frac{\sqrt{M|d|t}}{2\pi} \right) ight)$$

$$+ \left( \frac{L'_*}{L_*} \right)' \left( 1 + ir, (f \otimes \psi_d) \otimes (\overline{f} \otimes \psi_d) \right)$$

$$+ \frac{1}{c_{f \otimes \psi_d}^2} \left( \frac{\sqrt{M|d|t}}{2\pi} \right)^{-2ir} L(1 + ir, (f \otimes \psi_d) \otimes (\overline{f} \otimes \psi_d)) L(1 - ir, (f \otimes \psi_d) \otimes (\overline{f} \otimes \psi_d)) \mathcal{A}(ir)$$

$$- \mathcal{B}(1 + ir) \int dr \left| dt + O \left( T^{1/2 + \epsilon} \right) \right|,$$

where the inner integral is to be regarded as a principal value near $r = 0$, $f$ is new of level $M$, $M|d|^2 = N$, $\psi_d$ is the Kronecker character associated to a fundamental discriminant $d > 0$ prime to $M$, $c_{f \otimes \psi_d} = \text{res}_{s=1} L(s, (f \otimes \psi_d) \otimes (\overline{f} \otimes \psi_d))$ is an arithmetic constant, $\mathcal{A}(\rho)$ and $\mathcal{B}(1 + \rho)$ are Euler factors analytic near 1, and $L_*(s, (f \otimes \psi) \otimes (\overline{f} \otimes \psi_d))$ is the ‘unramified’ part of the Rankin-Selberg convolution.