A Characterization of Prime v-palindromes

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CANT, May 26, 2023

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$$198 = 2 \cdot 3^2 \cdot 11,$$
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and

$$2 + (3 + 2) + 11 = (3 + 4) + 11 = 18.$$

Introduction

Definition

Let $b \geq 2$, $L \geq 1$, and $0 \leq a_0, \ldots, a_{L-1} < b$ be integers. We denote

$$(a_{L-1} \cdots a_1 a_0)_b := \sum_{i=0}^{L-1} a_i b^i.$$

We write $(a_{L-1}, \ldots, a_1, a_0)_b$ to make it clear of which are each digit.

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Definition

Let the base $b \geq 2$ representation of an integer $n \geq 1$ be $(a_{L-1} \cdots a_1 a_0)_b$. The *$b$-reverse* of n is

$$r_b(n) := (a_0 a_1 \cdots a_{L-1})_b.$$

We write $r(n)$ for $r_{10}(n)$.

Introduction

Example

$$r(18) = 81,$$
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Define additive function $v\colon \mathbb{N} \to \mathbb{Z}$ by $v(p) := p$ for primes p and $v(p^\alpha) := p + \alpha$ for prime powers p^α with $\alpha \geq 2$.

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- If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1 \cdots q_m$, then

$$\begin{aligned} v(n) &= v(p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1 \cdots q_m) \\ &= v(p_1^{\alpha_1}) + \cdots + v(p_k^{\alpha_k}) + v(q_1) + \cdots + v(q_m) \\ &= (p_1 + \alpha_1) + \cdots + (p_k + \alpha_k) + q_1 + \cdots + q_m. \end{aligned}$$

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Let P be a statement. The *Iverson bracket* is defined by

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For integers $\alpha \geq 1$, denote $\iota(\alpha) := \alpha[\alpha > 1]$. That is,

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Define additive function $v\colon \mathbb{N} \to \mathbb{Z}$ by $v(p^\alpha) := p + \iota(\alpha)$ for prime powers p^α.

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Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then

$$v(n) = \sum_{i=1}^{k} (p_i + \iota(\alpha_i)), \qquad \text{(A338038)}$$

$$f(n) = \sum_{i=1}^{k} (p_i + \alpha_i), \qquad \text{(A008474)}$$

$$\mathrm{sopfr}(n) = \sum_{i=1}^{k} (p_i \alpha_i), \qquad \text{(A001414)}$$

$$g(n) = \prod_{i=1}^{k} (p_i \alpha_i). \qquad \text{(A000026)}$$

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An integer $n \geq 1$ is a *v-palindrome in base b* if $b \nmid n$, $n \neq r_b(n)$, and $v(n) = v(r_b(n))$. A v-palindrome in base 10 is simply called a *v-palindrome*.

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Example

- 576 is a v-palindrome because

$$576 = 2^6 \cdot 3^2, \quad r(576) = 675 = 3^3 \cdot 5^2,$$
$$v(576) = v(r(576)) = 13.$$

- Sequences of v-palindromes

$$18,\ 198,\ 1998,\ 19998,\ 199998,\ 1999998,\ \dots,$$
$$18, 1818, 181818, 18181818, 1818181818, 181818181818, \dots,$$

generalized by the following.

Introduction

Theorem (Tsai 2021)

If ρ is a decimal palindrome consisting entirely of 0's and 1's, then 18ρ is a v-palindrome.

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- There are infinitely many v-palindromes.
- The v-palindromes $n \leq 10^5$ with $n < r(n)$ are listed in our manuscript.
- The sequence of v-palindromes is A338039.

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- The sequence of v-palindromes is A338039.

Conjecture (anonymous 2018)

There are no prime v-palindromes.

Introduction

Main Theorem (Boran, Choi, Miller, Purice, and Tsai, 2022)

The prime v-palindromes are precisely the twin primes of the form

$$5 \cdot 10^m - 1 = 4\underbrace{9 \cdots 9}_{m},$$

for $m \geq 4$.

Introduction

Definition

Define function $b\colon \mathbb{N} \cup \{0\} \to \mathbb{Z}[\sqrt{2}]$ by $b(0) = 0$, $b(1) = 1$, and

$$\begin{aligned} b(3n) &= b(n), \\ b(3n+1) &= \sqrt{2}b(n) + b(n+1), \\ b(3n+2) &= b(n) + \sqrt{2}b(n+1), \end{aligned}$$

for $n \geq 0$.

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for $n \geq 0$.

Theorem (Spiegelhofer 2017)

- For all $n \geq 1$, we have $b(n) = b(r_3(n))$.
- Obtained family of (f, b) such that $f(n) = f(r_b(n))$ for all $n \geq 1$.

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An inequality

Definition

The number of decimal digits an integer $n \geq 1$ has is denoted by $L(n)$. By convention, $L(0) := 0$.

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Example

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Lemma

Let $n \geq 1$ be an integer. Then $v(n) \leq n$ and $L(v(n)) \leq L(n)$.

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Outline of the Proof

Assume that p is a prime v-palindrome with $L(p) = m + 1 \geq 5$. Then $p = v(p) = v(r(p))$.

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- If $r(p) < p$, then $v(r(p)) \leq r(p) < p$, so we must have $r(p) > p$.
- If $r(p)$ is prime, then $v(r(p)) = r(p) > p$, so $r(p)$ must be composite.

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- If $r(p)$ is prime, then $v(r(p)) = r(p) > p$, so $r(p)$ must be composite.
- Suppose that

$$r(p) = fq^{\beta},$$

where q is the largest prime factor of $r(p)$ and $q^{\beta} \parallel r(p)$. Let $L(q) = l$.

Outline of the Proof

Lemma

If $l \leq m - 1$, then

(i) $L(p - q - \beta) \geq m$,

(ii) $L(p - q - \iota(\beta)) \geq m$,

(iii) $L(v(f)) \geq m$.

Outline of the Proof

Lemma

We have

(i) $L(f) = m + 1 - L(q^\beta) + [r(p) < 10^{L(f)+L(q^\beta)-1}]$,

(ii) $L(q^\beta) = m + 1 - L(f) + [r(p) < 10^{L(f)+L(q^\beta)-1}]$,

(iii) $L(q^\beta) \geq l$,

(iv) $L(f) \leq m + 1$,

(v) $L(f) \leq m + 2 - l$,

(vi) $L(f) \leq m + 1 - \beta(l - 1)$, and

(vii) $L(p - q - \beta) \leq m + 2 - l$.

Outline of the Proof

- We divide $l \leq m+1$ into the cases
  - $l = 1$,
  - $l = 2$,
  - $3 \leq l \leq m-1$,
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  - $l = m+1$.
- Show that we must have $l = m+1$.
- Show further that $p = 5 \cdot 10^m - 1$ and $p - 2$ is also prime.

Outline of the Proof

Lemma

We cannot have $3 \leq l \leq m - 1$.

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We cannot have $3 \leq l \leq m-1$.

Proof.

Assume on the contrary that $3 \leq l \leq m-1$ is possible. Since $l \leq m-1$, by previous Lemmas,

$$m \leq L(p-q-\beta) \leq m+2-l.$$

This implies that $l \leq 2$, a contradiction. □

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Proof of the converse

Proof.

Let $p = 5 \cdot 10^m - 1 = 4\underbrace{9\cdots9}_{m}$, for some integer $m \geq 4$, be a prime such that $p-2$ is also prime. We show that p is a v-palindrome. Firstly, clearly $10 \nmid p$ and $p \neq r(p)$. We have

$$r(p) = r(4\underbrace{9\cdots9}_{m}) = \underbrace{9\cdots9}_{m}4 = 2 \cdot 4\underbrace{9\cdots9}_{m-1}7 = 2(p-2).$$

Consequently, as $p-2$ is an odd prime,

$$v(r(p)) = v(2(p-2)) = 2 + (p-2) = p.$$

This proves the converse. $\square$

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Number of prime v-palindromes

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- In a standard heuristic model, we assume that

$$\mathrm{Prob}(n \in \mathbb{P}) \ll \frac{1}{\log n}, \quad \mathrm{Prob}(n, m \in \mathbb{P}) \ll \frac{1}{\log n \cdot \log m}.$$

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- Let T_n be the event that $5 \cdot 10^n - 1, 5 \cdot 10^n - 3 \in \mathbb{P}$. Expected number of prime v-palindromes $\leq 10^{N+1}$ is

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- We over-estimate

$$\mathrm{Prob}(T_n) \ll \frac{1}{\log(5 \cdot 10^n - 3)} \frac{1}{\log(5 \cdot 10^n - 1)} \ll \frac{1}{n^2 \log^2 10} \ll \frac{1}{n^2}.$$

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- As $\sum 1/n^2$ converges, the expected number of prime v-palindromes is finite.

Number of prime v-palindromes

Conjecture

There are only finitely many prime v-palindromes.

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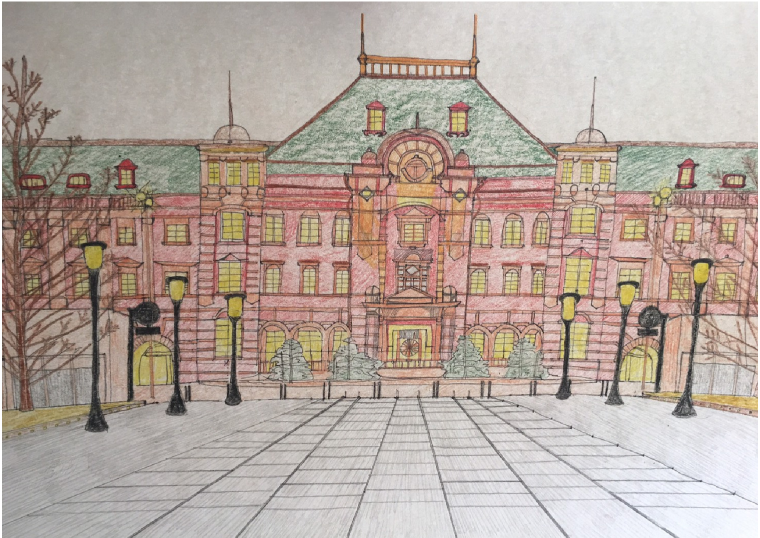
Thank you very much for listening!

This research was done as part of the Polymath Jr REU 2022 program.



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