Random Matrix Theory, Random Graphs, and L-Functions:

How the Manhatten Project helped us understand primes

Steven J. Miller

The Ohio State University

Cincinnati, October 16^{th}, 2003

http://www.math.ohio-state.edu/∼sjmiller/math/talks/talks.html

Collaborators

Random Matrices

- Peter Sarnak
- Rebecca Lehman
- Yi-Kai Liu
- Chris Hammond
- Inna Zakharevich

Random Graphs

- Peter Sarnak
- Yakov Sinai
- Lior Silberman
- Dustin Steinhauer
- Randy Qian
- Felice Kuan
- Leo Goldmakher

Fundamental Problem: Spacing Between Events

General Formulation: Studying some system, observe values at $t_1, t_2, t_3, \ldots$.

Question: what rules govern the spacings between events?

Often need to normalize by average spacing.

Examples:

- Spacings Between Energy Levels of Nuclei.

- Spacings Between Eigenvalues of Matrices.

- Spacings Between Zeros of L-Functions.

Eigenvalue Review

$$\begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$$

In general, in $A\overrightarrow{v} = \overrightarrow{w}$, $\overrightarrow{w}$ will have different **magnitude** and **direction** than $\overrightarrow{v}$.

$\overrightarrow{v}$ is an **eigenvector** with **eigenvalue** λ if

$$\begin{aligned} \overrightarrow{v} &\neq \overrightarrow{0} \\ A\overrightarrow{v} &= \lambda\overrightarrow{v}. \end{aligned}$$

Note

$$A^2\overrightarrow{v} = A(A\overrightarrow{v}) = A(\lambda\overrightarrow{v}) = \lambda^2\overrightarrow{v}.$$

Eigenvalue Review (cont)

Help us understand a matrix.

Say $\overrightarrow{v_i}$ eigenvectors with eigenvalues λ_i.

Assume

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}.$$

Then

$$\begin{aligned}
A^m\overrightarrow{v} &= A^m(c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}) \\
&= A^m(c_1\overrightarrow{v_1}) + \cdots + A^m(c_k\overrightarrow{v_k}) \\
&= c_1 A^m\overrightarrow{v_1} + \cdots + c_k A^m\overrightarrow{v_k} \\
&= c_1\lambda_1^m\overrightarrow{v_1} + \cdots + c_k\lambda_k^m\overrightarrow{v_k}.
\end{aligned}$$

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

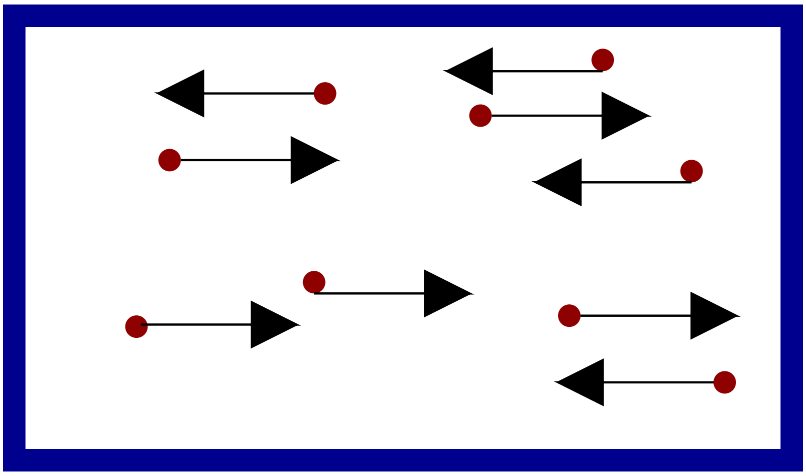
Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n = E_n\psi_n$$

E_n are the energy levels

Origins (cont)



Statistical Mechanics: for each configuration, calculate quantity (say pressure).

Average over all configurations – most configurations close to system average.

Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices.

Look at: Real Symmetric, Complex Hermitian, Classical Compact Groups.

Random Matrix Ensembles

Real Symmetric Matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{N} \end{pmatrix} = A^T$$

Let $p(x)$ be a probability density:

$$\begin{aligned} p(x) &\geq 0 \\ \int_{\mathbb{R}} p(x)dx &= 1. \end{aligned}$$

Often assume $p(x)$ has finite moments:

$$k^{th}\text{-moment} = \int_{\mathbb{R}} x^k p(x)dx.$$

Define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

Eigenvalue Distribution

Key to Averaging:

$$\text{Trace}(A^k) \;=\; \sum_{i=1}^{N} \lambda_i^k(A).$$

By the Central Limit Theorem:

$$\begin{aligned}
\text{Trace}(A^2) \;&=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \\
&=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \\
&\sim\; N^2 \cdot 1 \\
\sum_{i=1}^{N} \lambda_i^2(A) \;&\sim\; N^2
\end{aligned}$$

Gives $N\text{Ave}(\lambda_i^2(A)) \;\sim\; N^2$ or $\lambda_i(A) \sim \sqrt{N}$.

Eigenvalue Distribution (cont)

$\delta(x - x_0)$ is a unit point mass at x_0.

To each A, attach a probability measure:

$$\mu_{A,N}(x) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

Obtain:

$$\begin{aligned} k^{th}\text{-moment} \;&=\; \int x^k \mu_{A,N}(x)dx \\ &=\; \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i^k(A)}{(2\sqrt{N})^k} \\ &=\; \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \end{aligned}$$

Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$.

Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. Then

$$\mu_{A,N}(x) \;\rightarrow\; \frac{2}{\pi}\sqrt{1-x^2} \text{ with probability } 1$$

Trace formula converts sums over eigenvalues to sums over entries of A.

Expected value of k^{th}-moment of $\mu_{A,N}(x)$ is

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \prod_{i<j} p(a_{ij}) da_{ij}$$

Proof: 2^{nd}-Moment

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2.$$

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\sum_{i,j=1}^{N} a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

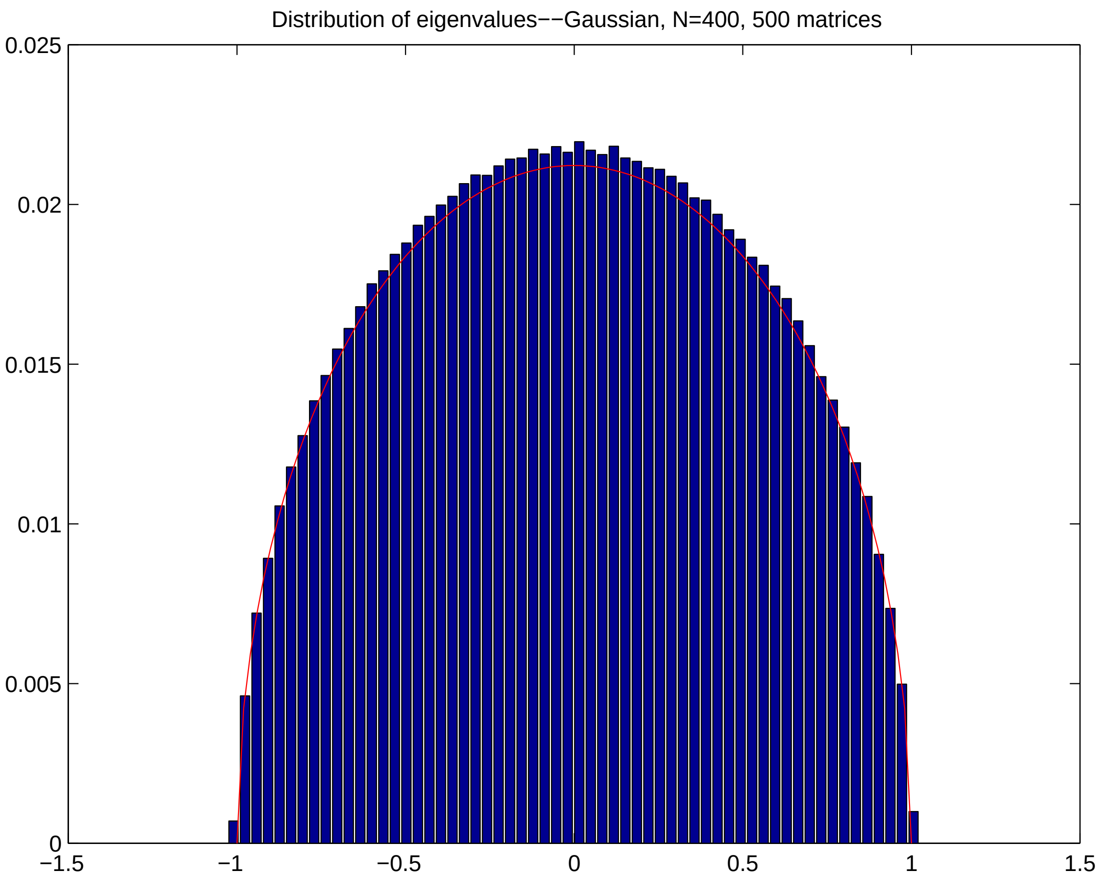
Integration factors as

$$\int_{a_{ij}\in\mathbb{R}} a_{ij}^2 p(a_{ij})da_{ij} \;\;\cdot \prod_{\substack{(k,l)\neq(ij)\\ k<l}} \int_{a_{kl}\in\mathbb{R}} p(a_{kl})da_{kl} \;=\; 1.$$

Have N^2 summands, answer is $\frac{1}{4}$.

Key: Averaging Formula.

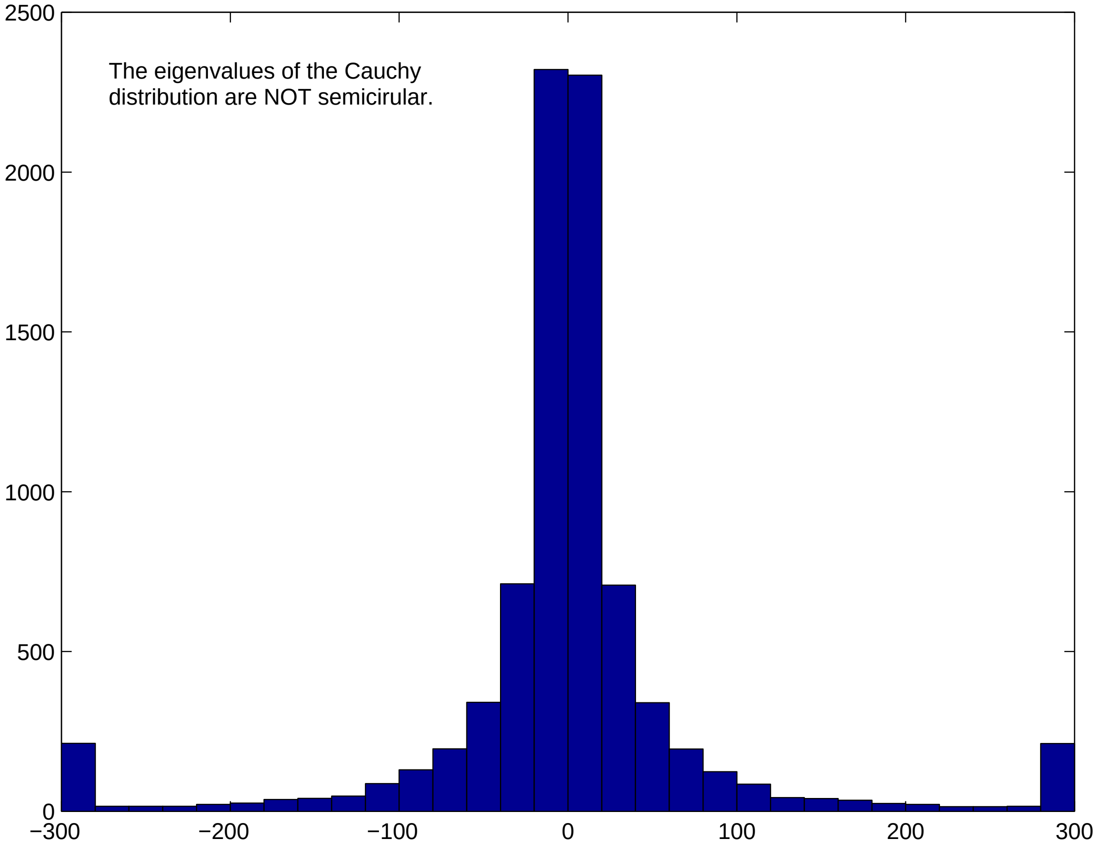
Random Matrix Theory: Semi-Circle Law



500 **Matrices: Gaussian** 400×400

$$p(x) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Random Matrix Theory:
Semi-Circle Law



Cauchy Distr: Not-Semicircular (Infinite Variance)

$$p(x) = \frac{1}{\pi(1+x^2)}$$

GOE Conjecture

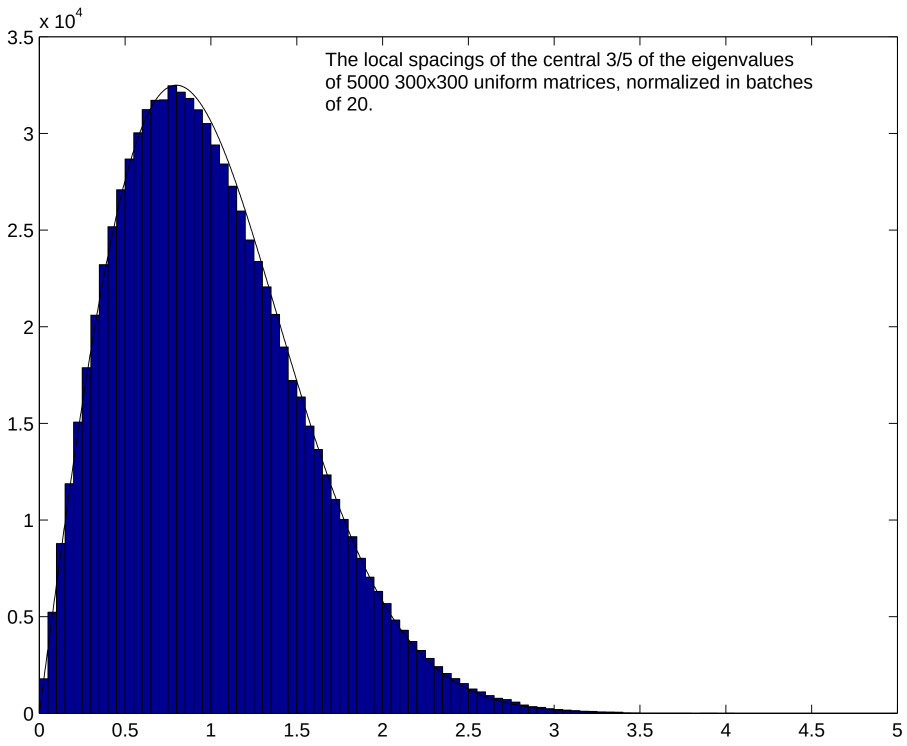
GOE Conjecture: As $N \to \infty$, the probability density of the distance between two consecutive, normalized eigenvalues approaches $\frac{\pi^2}{4}\frac{d^2\Psi}{dt^2}$ (the GOE distr).

$\Psi(t)$ is (up to constants) the Fredholm determinant of the operator $f \longrightarrow \int_{-t}^{t} K * f$, kernel $K = \frac{1}{2\pi}\left(\frac{\sin(\xi-\eta)}{\xi-\eta} + \frac{\sin(\xi+\eta)}{\xi+\eta}\right)$.

Only known if entries chosen from Gaussian.

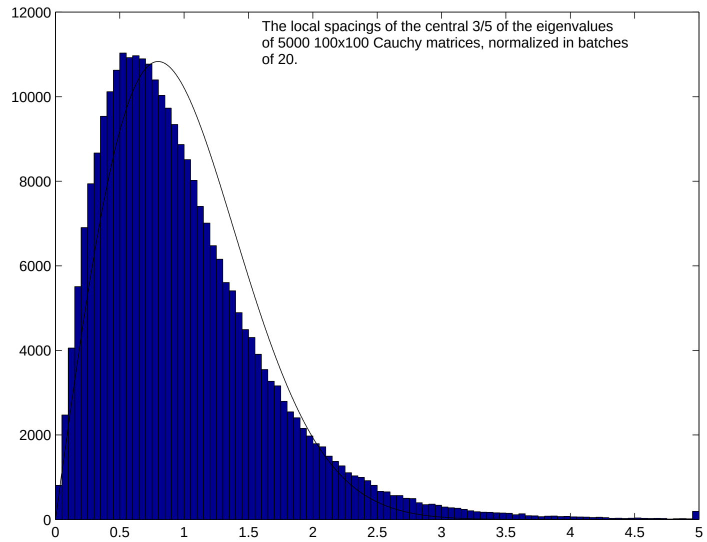
Consecutive spacings well approximated by Axe^{-Bx^2}.

Uniform Distribution: $p(x) = \frac{1}{2}$

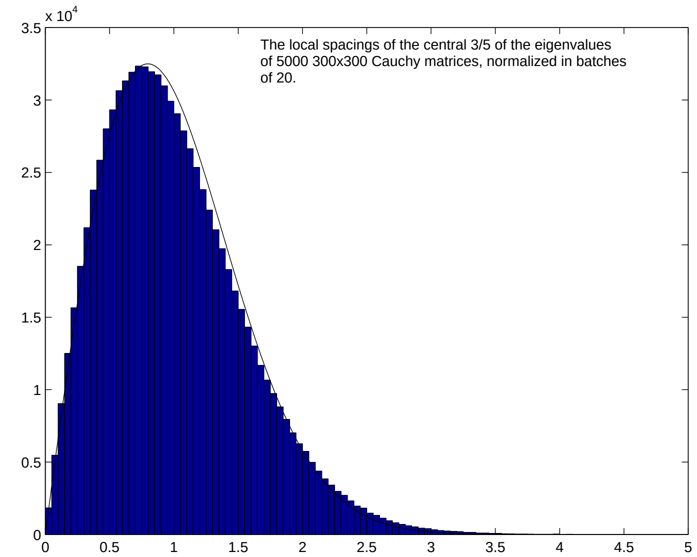


5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

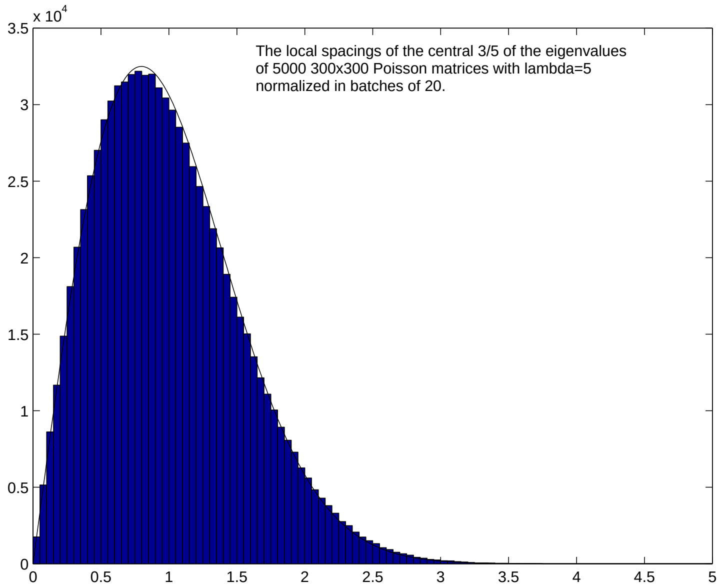


5000: 100×100 Cauchy

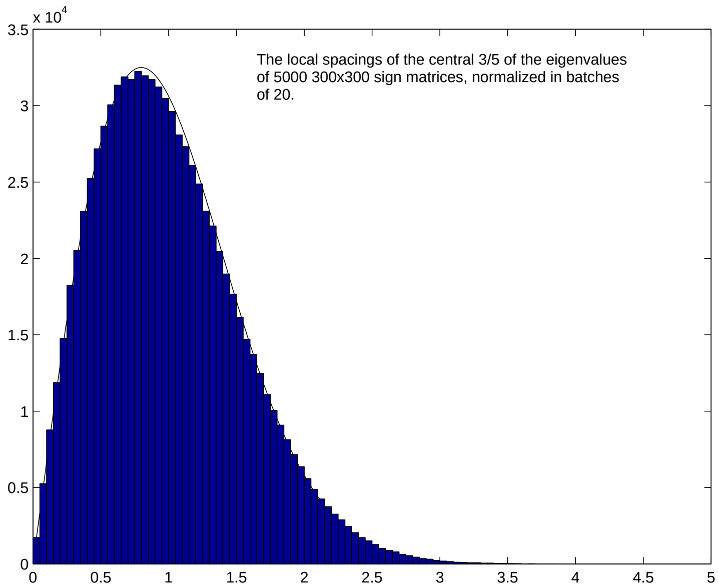


5000: 300×300 Cauchy

Poisson Distribution: $p(n) = \frac{\lambda^n}{n!} e^{-\lambda}$



5000: 300×300 Poisson, $\lambda = 5$



5000: 300×300 Poisson, $\lambda = 20$

Fat Thin Families

Need a family ***FAT*** enough to do averaging.

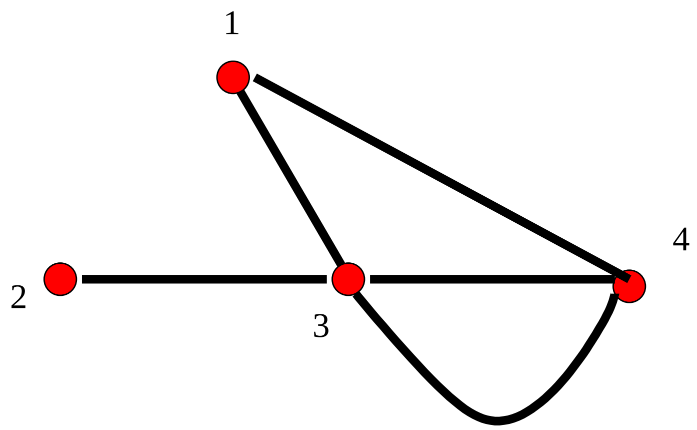
Need a family ***THIN*** enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

Examples of thin sub-families:

- Band Matrices
- Random Graphs
- Special Matrices (Toeplitz)

Random Graphs



Degree of a vertex = number of edges leaving the vertex.

Adjacency matrix: a_{ij} = number edges from Vertex i to Vertex j.

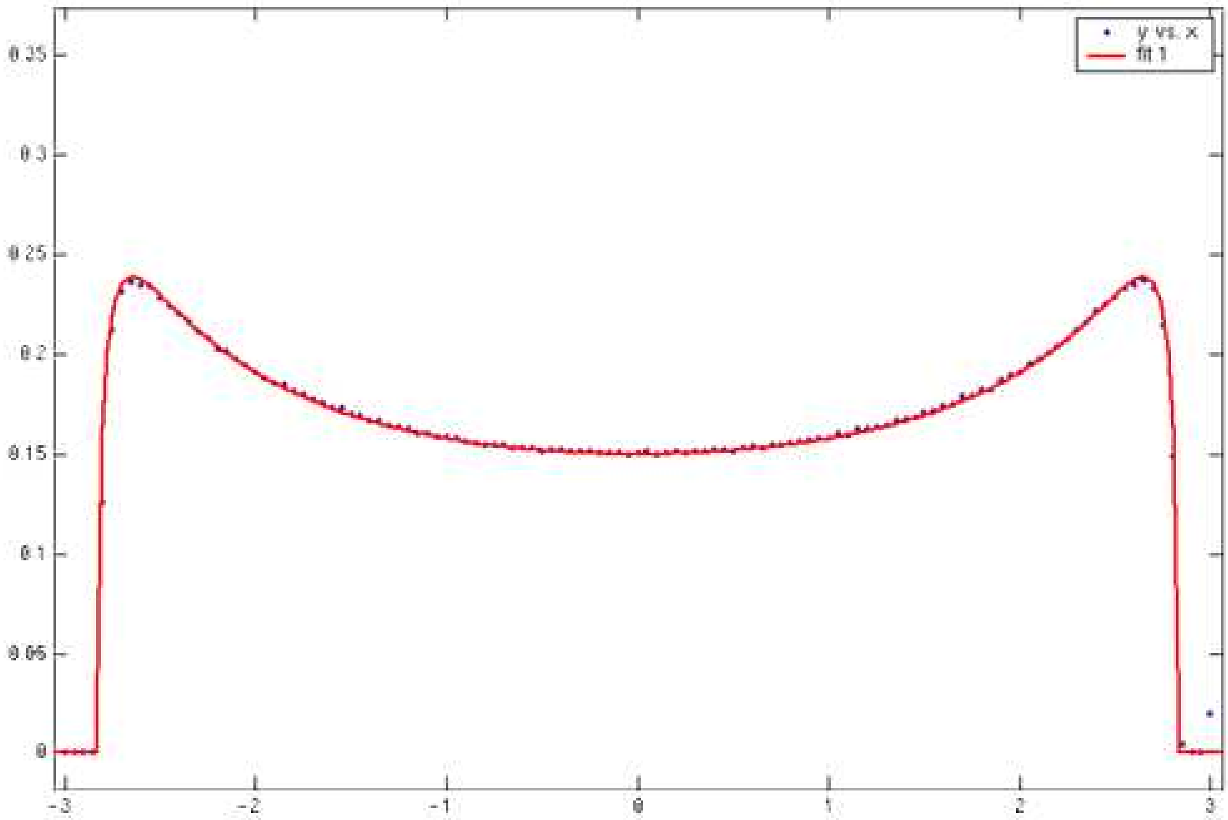
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

McKay's Law (Kesten Measure)

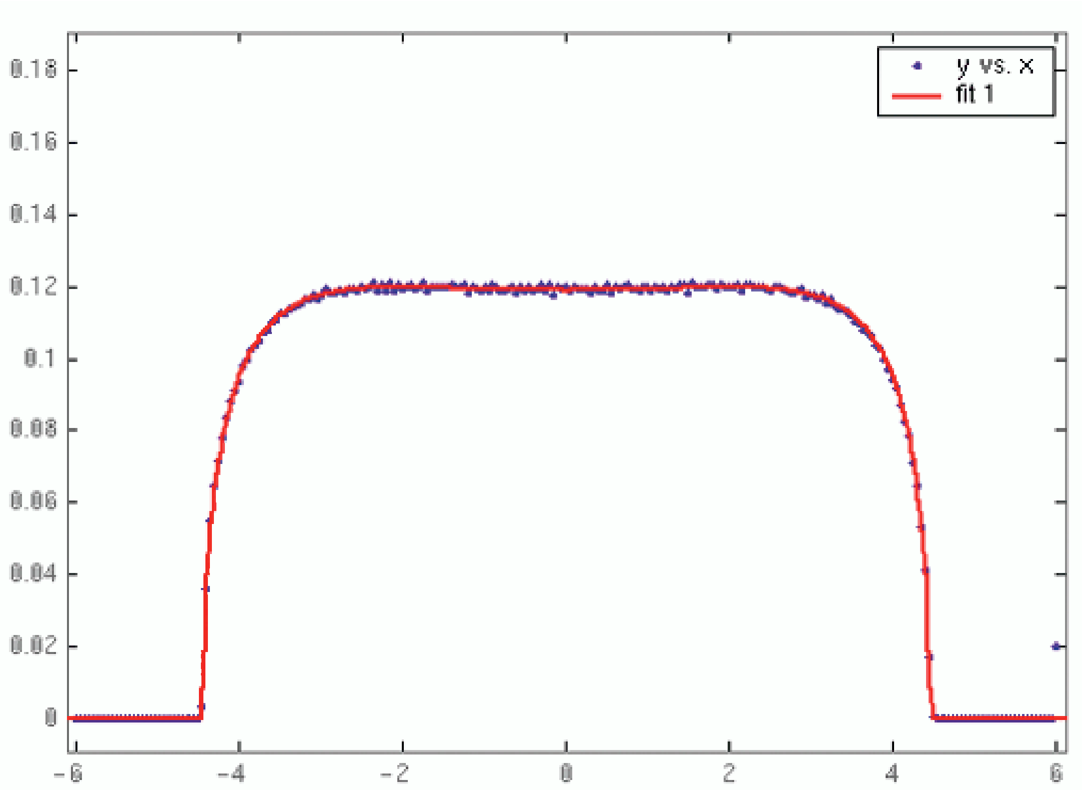
Density of States for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise} \end{cases}$$



$$d = 3.$$

McKay's Law (Kesten Measure)

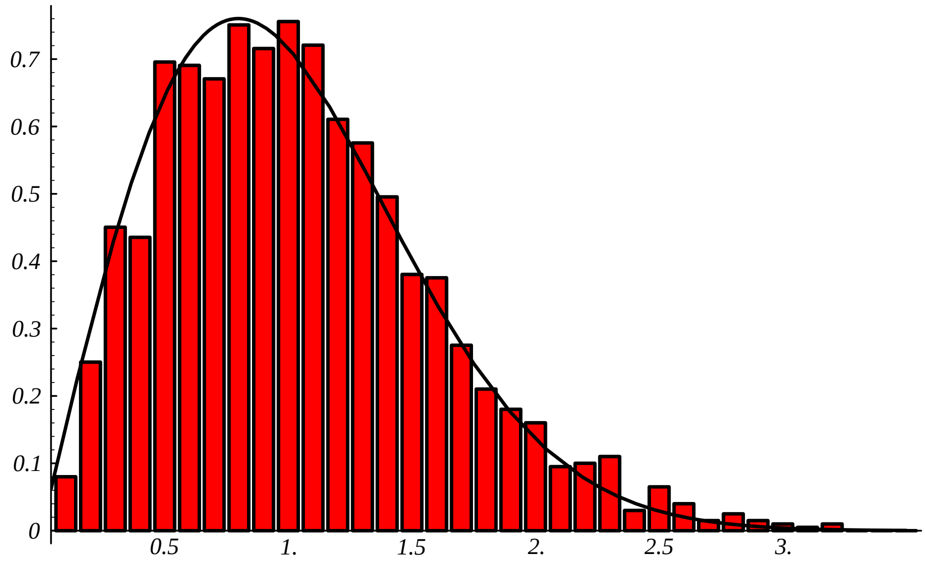


$$d = 6.$$

Idea of proof: Trace lemma, combinatorics and counting.

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

d-Regular and GOE



3-Regular, 2000 Vertices
Graph courtesy of D. Jakobson, S. D. Miller, Z. Rudnick, R. Rivin

Riemann Zeta Function: $\zeta(s)$

Riemann Zeta-Function:

$$\zeta(s) = \sum_n n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis: All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; ie, on the critical line.

Spacings between zeros same as spacings between eigenvalues of Complex Hermitian matrices.

Riemann Zeta Function: (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Geometric Series: If $|u| < 1$,

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots = \sum_{k=0}^{\infty} u^k.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function: (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$
$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$;
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$;
- Deep: GUE and arithmetic progressions.

Arithmetic Progression: $(a, b) = 1$, $an + b$.

Zero Knowledge
(Heuristic)

$P(x)$ polynomial, zeros $r_1, \ldots, r_N$.

Then

$$\begin{aligned} P(x) &= A \cdot (x - r_1)(x - r_2) \cdots (x - r_n) \\ &= A\Big(x^n + a_{n-1}(r_1, \ldots, r_n)x^{n-1} + \cdots \\ &\qquad \cdots + a_0(r_1, \ldots, r_n)\Big) \end{aligned}$$

where

$$\begin{aligned} a_{n-1}(r_1, \ldots, r_n) &= -(r_1 + \cdots + r_n) \\ &\vdots \\ a_0(r_1, \ldots, r_n) &= r_1 r_2 \cdots r_n. \end{aligned}$$

Knowledge of zeros gives info on coefficients.

Families of L-Functions

More generally, we may consider an L-function

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} \;=\; \prod_p L_p(p^{-s},f)^{-1}, \quad \text{Re}(s) > s_0.$$

Examples:

- Dirichlet Characters: $a_n(f) \;=\; \chi_f(n)$.
- Elliptic Curves: $y^2 = x^3 + A_f x + B_f$, $a_p(f)$ is related to number of solns mod p.

General Riemann Hypothesis: All L-functions (after normalization) have their zeros on the critical line.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ be an increasing sequence of numbers, $B \subset \mathbf{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i\neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

Results:

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. insensitive to any finite set of zeros

Measures of Spacings: n-Level Density and Families

Let $f(x) = \prod_i f_i(x_i)$, f_i even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \;=\; \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\left(L_E\gamma_E^{(j_1)}\right)\cdots f_n\left(L_E\gamma_E^{(j_n)}\right)$$

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the n-level density depends only on a symmetry group attached to the family.

Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
- Symplectic: Rubinstein: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.
- Unitary: Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
- Orthogonal: Miller: One-parameter families of elliptic curves.

Main Tools:

- Averaging Formulas: Petersson formula in ILS, Orthogonality of characters in Rubinstein, Miller, Hughes-Rudnick.
- Control of conductors: Monotone.

Correspondences

Similarities b/w Nuclei and L-Fns:

$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Support} \quad \longleftrightarrow \quad \text{Neutron Energy.}$$

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,O^+}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,O}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,O^-}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and $\eta(u)$ is 1, $\frac{1}{2}$, and 0 for $|u|$ less than 1, 1, and greater than 1.

Explicit Formula

Starting Point is the Explicit Formula, which relates sums of test functions over zeros to sums over primes.

For Elliptic Curves

$$\begin{aligned}\sum_{\gamma_E^{(j)}} G\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) &= \widehat{G}(0) + G(0) \\ &- 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log N_E}\right) a_E(p) \\ &- 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log N_E}\right) a_E^2(p) \\ &+ O\left(\frac{\log\log N_E}{\log N_E}\right).\end{aligned}$$

Ingredients of proof:

Complex Analysis (Shifting Contours)

Summary

- Similar behavior in different systems.

- Find correct scale.

- Average over similar elements.

- Need an Explicit Formula.

- Thin subsets can exhibit very different behavior.

- Different statistics tell different stories.