Workshop on spectral theory and automorphic forms

Random Matrix Theory and Families of Elliptic Curves: Evidence for the Underlying Group Symmetries

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Additional comments not on slides for talk added in these notes.

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Info by shooting high-energy neutrons into nucleus.

Fundamental Equation: Quantum Mechanics

$$H\psi_n \ = \ E_n\psi_n$$

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric (GOE), Complex Hermitian (GUE), Classical Compact Groups.
First two: say choose entries iidrv from fixed probability distribution p; however, no unique choice of p. Third choose matrices wrt Haar; have a canonical choice of random.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k \leq N\right\}}{N}$$

Observations and Results:

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. **insensitive to any finite set of zeros**

Measures of Spacings: n-Level Density and Families

$\phi(x) = \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions, $\widehat{\phi}_i$ compactly supported.

$$D_{n,f}(\phi) \;=\; \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

L_f = Conductor, the scale factor for low zeros.

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

$$D_{n,\mathcal{F}}(\phi) \;=\; \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_{n,f}(\phi).$$

Limiting Behavior

As $N \to \infty$,

$$\begin{aligned} & \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \\ = & \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\gamma_f^{(j_i)} \log L_f}{2\pi}\right) \\ \to & \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx \\ \to & \int \cdots \int \widehat{\phi}(y) \widehat{W}_{n,\mathcal{G}(\mathcal{F})}(y) dy. \end{aligned}$$

Conj: Distribution of Low Zeros agrees with a classical compact group.

Correspondences

Similarities b/w Nuclear Physics and L-Functions:

$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Support} \quad \longleftrightarrow \quad \text{Neutron Energy}.$$

Conjecture: Zeros near central point in a family of L-functions behave like eigenvalues near 1 of a classical compact group (Unitary, Symplectic, Orthogonal).

Some Number Theory Results

- Orthogonal:

  Iwaniec-Luo-Sarnak: 1-level density for $H_k^{\pm}(N)$, N square-free;

  Dueñez-Miller: 1, 2-level $\{\phi \times \text{sym}^2 f : f \in H_k(1)\}$, ϕ even Maass;

  Miller: 1, 2-level for one-parameter families of elliptic curves.

- Symplectic:

  Rubinstein: n-level densities for $L(s, \chi_d)$;

  Dueñez-Miller: 1-level for $\{\phi \times f : f \in H_k(1)\}$, ϕ even Maass.

- Unitary:

  Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.

Main Tools

- Explicit Formula: Relates sums over zeros to sums over primes.

- Averaging Formulas: Orthogonality of characters, Petersson formula.

- Control of conductors: Monotone.

1-Level Densities

Fourier Transforms for 1-level densities:

$$\begin{aligned}
\widehat{W}_{1,SO(\text{even})}(u) &= \delta(u) + \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,O}(u) &= \delta(u) + \tfrac{1}{2} \\
\widehat{W}_{1,SO(\text{odd})}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) + 1 \\
\widehat{W}_{1,Sp}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,U}(u) &= \delta(u)
\end{aligned}$$

where $\delta(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

For arbitrarily small support, can mutually distinguish unitary from symplectic from the three orthogonals; however, the three orthogonals are indistinguishable for support in $(-1,1)$.

Motivation: Dirichlet Characters: m Prime

$\{\chi_0\} \cup \{\chi_l\}_{l \le m-2}$ are all the characters mod m.

Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned}
\frac{1}{m-2}\sum_{\chi \neq \chi_0}\sum_{\gamma_\chi} \phi\left(\gamma_\chi \frac{\log(\frac{m}{\pi})}{2\pi}\right) &= \int_{-\infty}^{\infty} \phi(y)dy \\
&- \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p)+\bar{\chi}(p)]p^{-\frac{1}{2}} \\
&- \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p)+\bar{\chi}^2(p)]p^{-1} \\
&+ O\left(\frac{1}{\log m}\right).
\end{aligned}$$

Can pass Character Sum through Test Function.

Agrees with unitary for support in $(-2, 2)$. See Hughes-Rudnick, S. J. Miller.

Elliptic Curves

Conductors grow rapidly.

Results for small support, where Orthogonal densities indistinguishable.

Study 1 and 2-Level Densities.

$$D_{n,f}(\phi) \ = \ \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\Big(L_f\gamma_f^{(j_1)}\Big)\cdots\phi_n\Big(L_f\gamma_f^{(j_n)}\Big)$$

$$D_{n,\mathcal{F}}(\phi) \ = \ \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} D_{n,f}(\phi).$$

2-Level Densities

$$c(\mathcal{G}) = \begin{cases} 0 & \text{if } \mathcal{G} = \text{SO(even)} \\ \frac{1}{2} & \text{if } \mathcal{G} = \text{O} \\ 1 & \text{if } \mathcal{G} = \text{SO(odd)} \end{cases}$$

For $\mathcal{G} = \text{SO(even)}, \text{O}$ or SO(odd):

$$\begin{aligned} \int\int \widehat{\phi}_1(u_1)\widehat{\phi}_2(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 &= \left[\widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0)\right]\left[\widehat{f}_2(0) + \frac{1}{2}\phi_2(0)\right] \\ &+ 2\int |u|\widehat{\phi}_1(u)\widehat{\phi}_2(u)du \\ &-2\widehat{\phi_1\phi_2}(0) - \phi_1(0)\phi_2(0) \\ &+ c(\mathcal{G})\phi_1(0)\phi_2(0). \end{aligned}$$

For arbitrarily small support, 2-level density can distinguish the three orthogonal groups.

Elliptic Curves

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{Q}$$

Often can write $E : y^2 = x^3 + Ax + B$.

Let N_p be the number of solns mod p:

$$N_p \;=\; \sum_{x(p)} \left[1 + \left(\frac{x^3 + Ax + B}{p}\right)\right] \;=\; p + \sum_{x(p)} \left(\frac{x^3 + Ax + B}{p}\right)$$

Local data: $a_E(p) = p - N_p$. Use to build the L-function:

$$a_E(p) \;=\; -\sum_{x \bmod p} \left(\frac{x^3 + Ax + B}{p}\right)$$

Elliptic Curves: Arithmetic Progression

One-parameter families:

$$E_t : y^2 \;=\; x^3 + A(t)x + B(t), \quad A(t), B(t) \in \mathbb{Z}(t).$$

We have

$$a_t(p) \;=\; -\sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p}\right) \;=\; a_{t+mp}(p)$$

Can handle sums of $a_t(p)$ for t in arithmetic progression.

Elliptic Curves (cont)

$$L(E,s) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_p L_p(E,s).$$

By GRH: All zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank r equals analytic rank (order of vanishing at central point).

Comments on Previous Results

- explicit formula relating zeros and Fourier coeffs;
- averaging formulas for the family;
- conductors easy to control (constant or monotone)

Elliptic curve E_t: discriminant $\Delta(t)$, conductor $N_{E_t} = C(t)$ is

$$C(t) \;=\; \prod_{p|\Delta(t)} p^{f_p(t)}$$

Normalization of Zeros

Local (hard) vs Global (easy). As $N \to \infty$:

$$\frac{1}{|\mathcal{F}_N|}\sum_{E\in\mathcal{F}} D_{n,E}(\phi) \;=\; \frac{1}{|\mathcal{F}_N|}\sum_{E\in\mathcal{F}}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$\to \int\cdots\int \widehat{\phi}(y)\widehat{W}_{n,\mathcal{G}(\mathcal{F})}(y)dy.$$

Conj: Distribution of Low Zeros agrees with Orthogonal Densities.

1-Level Expansion

$$
\begin{aligned}
D_{1,\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j}\phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) \\
&= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{\phi}(0)+\phi_i(0) \\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log N_E}\right)a_E(p) \\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{\phi}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p)+O\left(\frac{\log\log N_E}{\log N_E}\right)
\end{aligned}
$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$, leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

2-Level Expansion

Need to evaluate terms like

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\frac{1}{p_i^{r_i}}g_i\left(\frac{\log p_i}{\log N_E}\right)a_E^{r_i}(p_i).$$

Analogue of Petersson / Orthogonality: If $p_1,\dots,p_n$ are distinct primes

$$\sum_{t \bmod p_1\cdots p_n} a_t^{r_1}(p_1)\cdots a_t^{r_n}(p_n) \ = \ A_{r_1,\mathcal{F}}(p_1)\cdots A_{r_n,\mathcal{F}}(p_n).$$

Straightforward proof, but crucial observation. Without this, would have to handle sums of products; this reduces everything to "one-dimensional" sums.

Input

For many families

$$(1) : A_{1,\mathcal{F}}(p) \ = \ -rp \ + \ O(1)$$
$$(2) : A_{2,\mathcal{F}}(p) \ = \ p^2 \ + \ O(p^{3/2})$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(t)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p \le X} -\frac{A_{1,\mathcal{F}}(p) \log p}{p} = r$$

Surfaces with $j(t)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

DEFINITIONS

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$D_{n,\mathcal{F}}^{(r)}(\phi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(t)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

ϕ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (M–): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\phi) \longrightarrow \int \phi(x) W_{\mathcal{G}}(x) dx,$$

where

$$\mathcal{G} = \begin{cases} \text{O} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Of course, if the conductors are of size N^d, we expect first zero at about $\frac{c}{\log N}$. If instead of r zeros at the central point we had r zeros at height $\frac{1}{\log^{10000} N}$, could never notice the difference b/c of our error terms. Thus, as we never will be able to get infinite support (ie, letting ϕ be a Dirac delta functional), can never prove have r zeros at central point. But does provide evidence to support this.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free: $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   t^2+3t+9 Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(t)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$
\begin{aligned}
y^2 &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\
&\qquad + (2ct - D)(t^2 + 2t - A + 1)^2
\end{aligned}
$$

$$
\begin{aligned}
A &= 8,916,100,448,256,000,000 \\
B &= -811,365,140,824,616,222,208 \\
C &= 26,497,490,347,321,493,520,384 \\
D &= -343,107,594,345,448,813,363,200 \\
a &= 16,660,111,104 \\
b &= -1,603,174,809,600 \\
c &= 2,149,908,480,000
\end{aligned}
$$

Need GRH, Sq-Free Sieve to handle sieving.

Family looks more natural written in non-Weierstrass form. These are not smallest constants one can take. Example of a new method to find moderate rank families, what I call the "Discriminant Method".

Sketch of Proof

1. Sieving (Arithmetic Progressions)

2. Partial Summation (Complete Sums)

3. Controlling Conductors (Monotone).

Sieving

$$\sum_{\substack{t=N\\ D(t)\text{sqfree}}}^{2N} S(t) = \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\ t\in[N,2N]}} S(t)$$

$$= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\ t\in[N,2N]}} S(t) + \sum_{d\geq \log^l N}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\ t\in[N,2N]}} S(t).$$

Handle first by progressions.

Handle second by Cauchy-Schwartz:

The number of t in the second sum (by Sq-Free Sieve Conj) is $o(N)$:

Sieving (cont)

$$\sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\ t\in[N,2N]}} S(t)$$

$t_i(d)$ roots of $D(t) \equiv 0 \bmod d^2$.

$t_i(d), t_i(d) + d^2, \ldots, t_i(d) + \left[\frac{N}{d^2}\right] d^2$.

If $(d, p_1 p_2) = 1$, go through complete set of residue classes $\frac{N/d^2}{p_1 p_2}$ times.

As $d < \log^l N$, $l < 2$, can take all $p_i > \log^l N$ in the Explicit Formula, incorporating lower p_i's into the error terms.

Partial Summation

$\widetilde{a}_{d,i,p}(t') = a_{t(d,i,t')}(p)$, $G_{d,i,P}(u)$ is related to the test functions, d and i from progressions.

Applying Partial Summation

$$S(d,i,r,p) = \sum_{t'=0}^{[N/d^2]} \widetilde{a}^{r}_{d,i,p}(t') G_{d,i,p}(t')$$

$$= \left(\frac{[N/d^2]}{p} A_{r,\mathcal{F}}(p) + O\left(p^R\right) \right) G_{d,i,p}([N/d^2])$$

$$- \sum_{u=0}^{[N/d^2]-1} \left(\frac{u}{p} A_{r,\mathcal{F}}(p) + O\left(p^R\right) \right) \left(G_{d,i,p}(u) - G_{d,i,p}(u+1) \right)$$

Three terms straightforward, fourth difficult; will just discuss that one.

Difficult Piece: Fourth Sum I

$$\sum_{u=0}^{[N/d^2]-1} O(P^R)\Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big)$$

Taylor $G_{d,i,P}(u) - G_{d,i,P}(u+1)$ gives $P^R \frac{N}{d^2} \frac{1}{P^r \log N}$.

$\frac{1}{|\mathcal{F}|}\sum_{i,d}$ gives $O(\frac{P^R}{P^r \log N})$.

Problem is in summing over the primes, as we no longer have $\frac{1}{|\mathcal{F}|}$.

Problem is even if conductors range from $\log N^m$ to $\log 2N^m$, can't handle. Have $\frac{N}{d^2}$ terms, too many terms and bound depends on N.

Fourth Sum: II

If exactly one of the r_j's is non-zero, then

$$\begin{aligned}
&\sum_{u=0}^{[N/d^2]-1} \left| G_{d,i,P}(u) - G_{d,i,P}(u+1) \right| \\
= &\sum_{u=0}^{[N/d^2]-1} \left| g\left(\frac{\log p}{\log C(t_i(d)+ud^2)}\right) - g\left(\frac{\log p}{\log C(t_i(d)+(u+1)d^2)}\right) \right|
\end{aligned}$$

If conductors monotone, for fixed i, d and p, small independent of N (bounded variation).

This is the key observation: still have $\frac{N}{d^2}$ terms, but now by bounded variation (as long as ϕ continuously differentiable okay) the bound is independent of N – we have partitioned $[0, 2\sigma]$ into many points, but bound independent of number of partitions.

If two of the r_j's are non-zero:

$$\begin{aligned}
|a_1a_2 - b_1b_2| &= |a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2| \\
&\leq |a_1a_2 - b_1a_2| + |b_1a_2 - b_1b_2| \\
&= |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2|
\end{aligned}$$

Handling the Conductors: I

$$y^2 + a_1(t)xy + a_3(t)y \;=\; x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

$$C(t) \;=\; \prod_{p|\Delta(t)} p^{f_p(t)}$$

$$D_1(t) \;=\; \text{primitive irred poly factors } \Delta(t) \text{ and } c_4(t) \text{ share}$$

$$D_2(t) \;=\; \text{remaining primitive irred poly factors of } \Delta(t)$$

$$D(t) \;=\; D_1(t)D_2(t)$$

$D(t)$ sq-free, $C(t)$ like $D_1^2(t)D_2(t)$ except for a finite set of bad primes.

Handling the Conductors: II

$$y^2 + a_1(t)xy + a_3(t)y \;=\; x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

Let P be the product of the bad primes.

Tate's Algorithm gives $f_p(t)$, depend only on $a_i(t)$ mod powers of p.

Apply Tate's Algorithm to E_{t_1}. Get $f_p(t_1)$ for $p|P$. For m large, $p|P$,

$$f_p(\tau) = f_p(P^m t + t_1) = f_p(t_1),$$

and order of p dividing $D(P^m t + t_1)$ is independent of t.

Get integers st $C(\tau) = c_{bad}\frac{D_1^2(\tau)}{c_1}\frac{D_2(\tau)}{c_2}$, $D(\tau)$ sq-free.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(t)$, RMT $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r	= 32%
Percent with rank r+1	= 48%
Percent with rank r+2	= 18%
Percent with rank r+3	= 2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$?

M. Watkins has recently seen the excess rank is decreasing in the family $^3 + y^3 = m$.

Orthogonal Random Matrix Model

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j$$

Model: forced zeros are independent

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} g & \\ & I_{2r} \end{pmatrix} : g \in SO(2N-2r) \right\}$$

Another possible model is to look at the joint eigenvalue density, conditional on $2r$ eigenvalues equal 1. Eduardo Dueñez has done this calculation; however, the function field suggests the block diagonal model. This second model demonstrates repulsion of eigenvalues.

Testing RMT Model

For small support, 1-level densities for Elliptic Curves agree with $\rho_{r,\text{Indep}}$.

Curve E, conductor N_E, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log N_E}$.

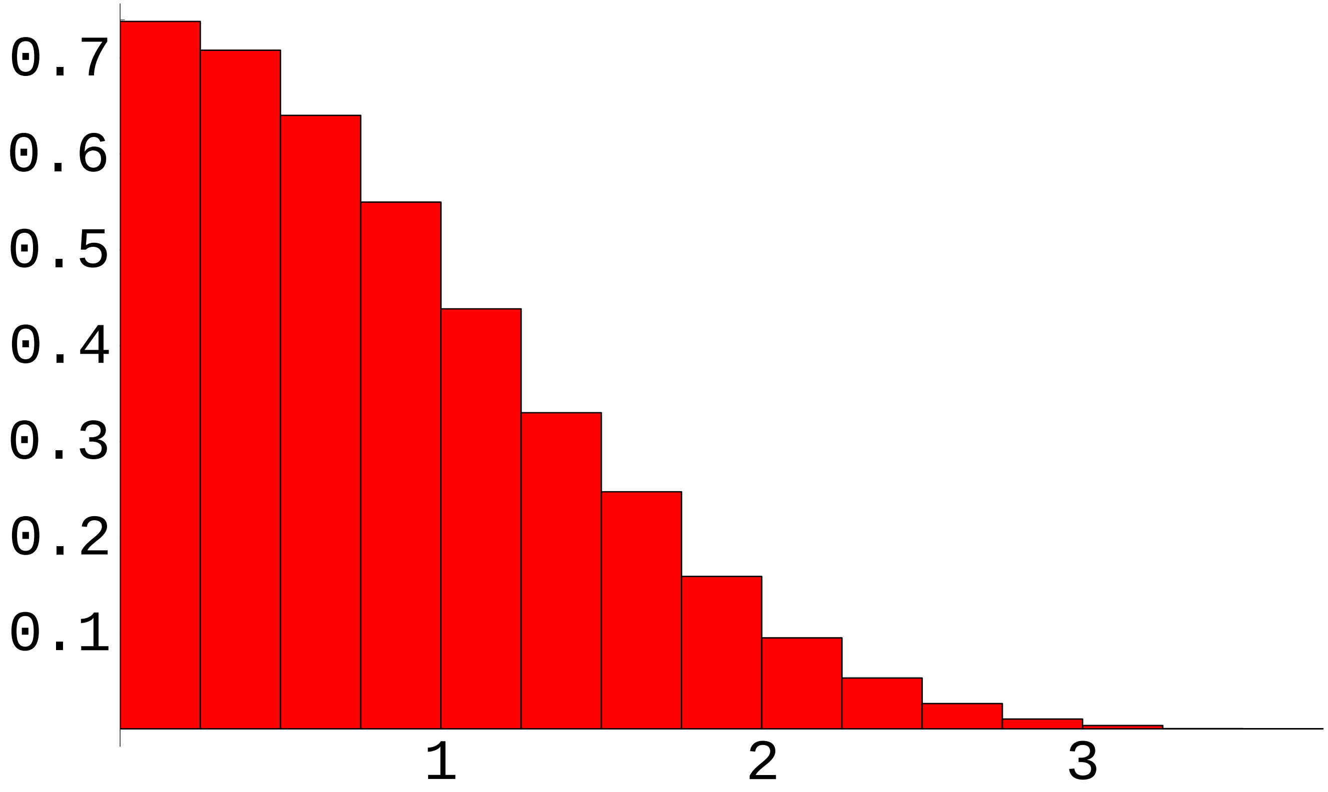
If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log N_E}$, might detect in 1-level density:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_E^{(j)} \log N_E}{2\pi}\right).$$

Corrections of size

$$\phi\left(x_0 + c_r\right) - \phi(x_0) \ \approx \ \phi'\left(x(x_0, c_r)\right) \cdot c_r.$$

Theoretical Distribution of First Normalized Zero

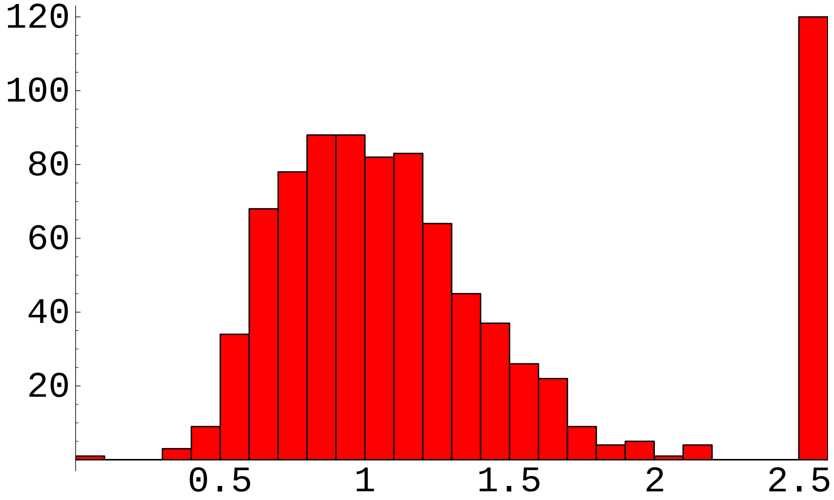


First normalized eigenvalue: 96000 from $\mathrm{SO}(4)$ with Haar Measure

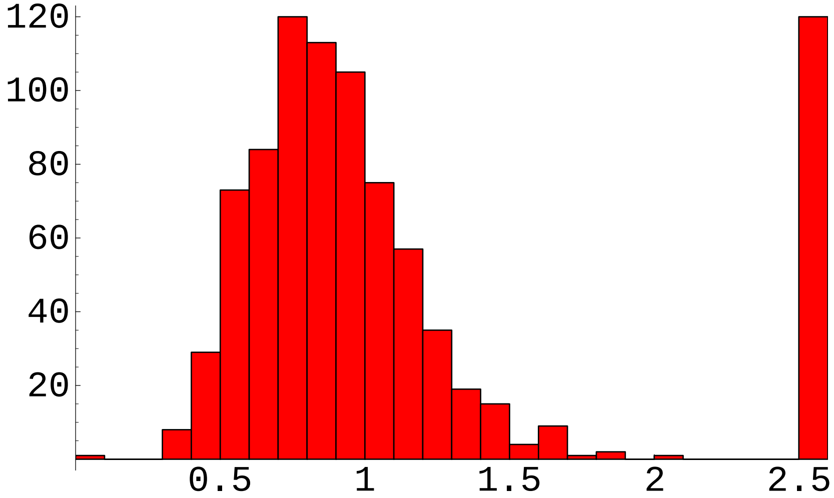
Easier to plot this than the Fredholm determinants, gives general shape. Eigenvalues near 1 VERY probable, no repulsion.

Rank 0 Curves: 1st Normalized Zero

(Far left and right bins just for formatting)

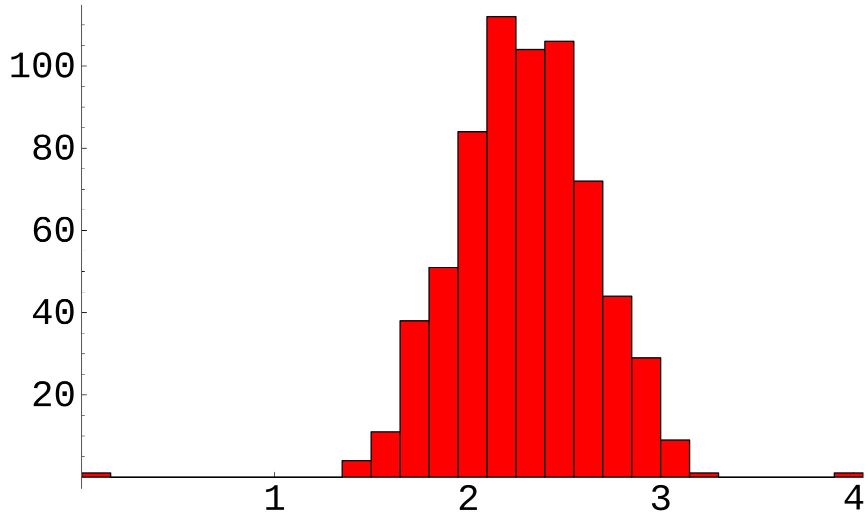


750 curves, $\log(\text{cond}) \in [3.2, 12.6]$; mean $= 1.04$

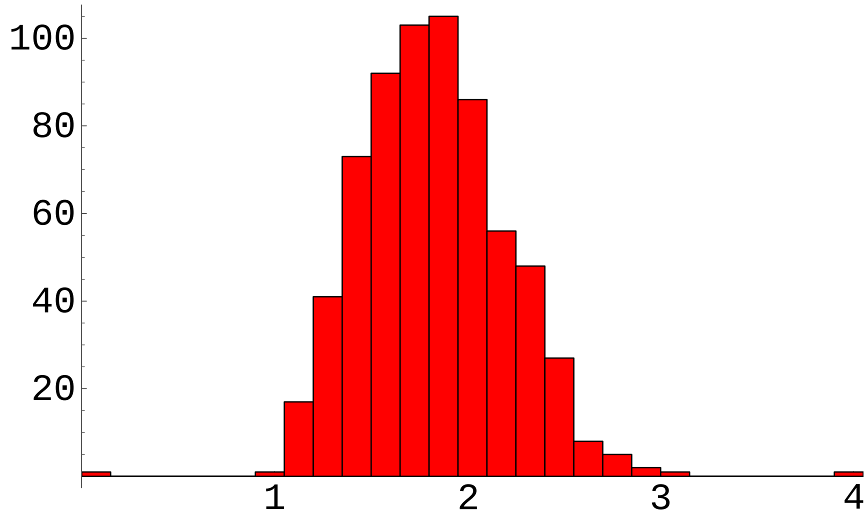


750 curves, $\log(\text{cond}) \in [12.6, 14.9]$; mean $= .88$

Rank 2 Curves: 1st Normalized Zero

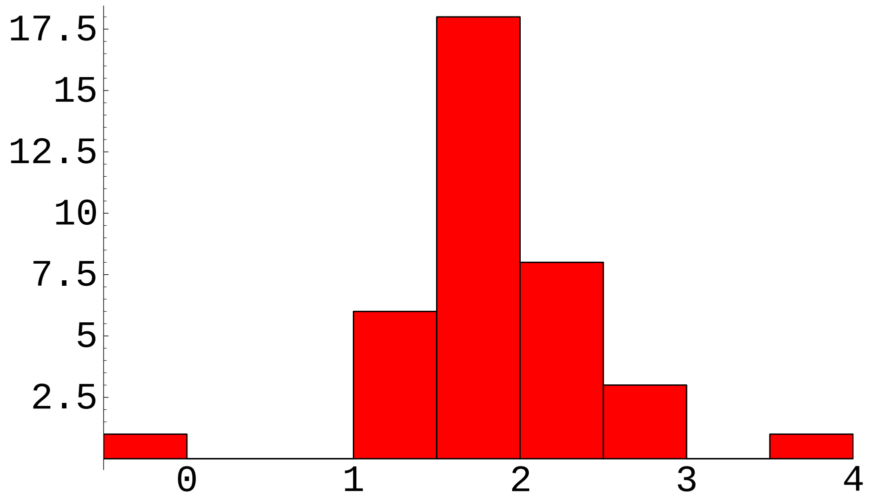


665 curves, $\log(\text{cond}) \in [10, 10.3125]$; mean $= 2.30$

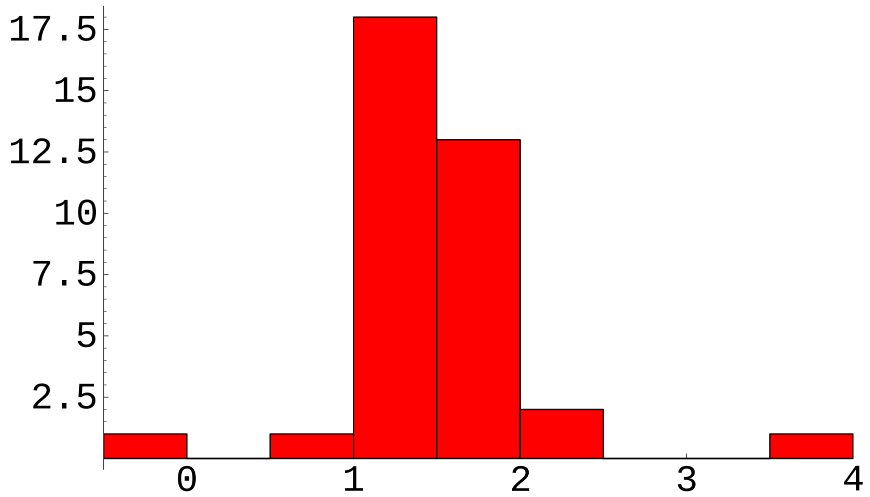


665 curves, $\log(\text{cond}) \in [16, 16.5]$; mean $= 1.82$

Rank 2 Curves: $[0,0,0,-t^2,t^2]$ 1st Normalized Zero



35 curves, $\log(\text{cond}) \in [7.8, 16.1]$; mean $= 2.24$



34 curves, $\log(\text{cond}) \in [16.2, 23.3]$; mean $= 2.00$

Comments on Histogram Slides

[NOT A SLIDE, FROM TALK]
Like excess rank, the data is not good; theory predicts in the limit there should be no repulsion from the additional zeros at the central point. Not only are we seeing repulsion when we have rank 2, we are seeing it when we have rank 0 curves!

Possible explanations: the function field analogies give the correct model in the limit, but could be poor for finite conductors. For the larger data sets, we are amalgamating several one-parameter families (similar to physicists combining experiments from several different heavy nuclei). We are, however, in a better position than excess rank – even for our small ranges, we do see the repulsion decrease as we increase the conductors.

Summary

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need an Explicit Formula.
- Different statistics tell different stories.
- Evidence for B-SD, RMT interpretation of zeros
- Need more data.

Slides on Excess Rank

The following slides were not presented in the talk proper, but were presented during questions. The code used to calculate the rank the central point is optimized for small conductors; at the cost of more programming, the code could be slightly improved.

Data on Excess Rank

$$y^2 + a_1 xy + a_3 y \;=\; x^3 + a_2 x^2 + a_4 x + a6$$

Family: $a_1 : 0$ to 10, rest -10 to 10.

Percent with rank 0 = 28.60%
Percent with rank 1 = 47.56%
Percent with rank 2 = 20.97%
Percent with rank 3 = 2.79%
Percent with rank 4 = .08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Data on Excess Rank

$$y^2 + y = x^3 + tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{11}, but on logarithmic scale still small.

Excess Rank Calculations
Families with $y^2 = f_t(x)$; $D(t)$ SqFree

Family	t Range	Num t	r	r	$r+1$	$r+2$	$r+3$
$+4(4t+2)$	$[2, 2002]$	1622	0		95.44		4.56
$-4(4t+2)$	$[2, 2002]$	1622	0	70.53		29.35	
$9t+1$	$[2, 247]$	169	0	71.01		28.99	
t^2+9t+1	$[2, 272]$	169	1	71.60		27.81	
$t(t-1)$	$[2, 2002]$	643	0	40.44	48.68	10.26	0.62
$(6t+1)x^2$	$[2, 101]$	93	1	34.41	47.31	17.20	1.08
$(6t+1)x$	$[2, 77]$	66	2	30.30	50.00	16.67	3.03

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.
2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.
4. $x^3 + tx^2 - (t+3)x + 1$, t^2+3t+9 Sq-Free, odd.
5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank 0.
6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank 1.
7. $x^3 - (6t+1)^2x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank 2.

Excess Rank Calculations
Families with $y^2 = f_t(x)$; All $D(t)$

Family	t Range	Num t	r	r	$r+1$	$r+2$	$r+3$
$+4(4t+2)$	$[2, 2002]$	2001	0	6.45	85.76	3.95	3.85
$-4(4t+2)$	$[2, 2002]$	2001	0	63.52	9.90	25.99	.50
$9t+1$	$[2, 247]$	247	0	55.28	23.98	20.73	
t^2+9t+1	$[2, 272]$	271	1	73.80		25.83	
$t(t-1)$	$[2, 2002]$	2001	0	42.03	48.43	9.25	0.30
$(6t+1)x^2$	$[2, 101]$	100	1	32.00	50.00	17.00	1.00
$(6t+1)x$	$[2, 77]$	76	2	32.89	50.00	14.47	2.63

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.

2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.

3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.

4. $x^3 + tx^2 - (t+3)x + 1$, t^2+3t+9 Sq-Free, odd.

5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank 0.

6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank 1.

7. $x^3 - (6t+1)^2 x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank 2.

Appendices

The first two appendices list various standard conjectures. The second provides (at least conjecturally) when a family should have equidistribution of signs of functional equations. Experimental evidence is provided in the third appendix, which is on the distribution of signs of elliptic curves in a one-parameter family. Testing whether or not a generic family is equidistributed in sign. We looked at 1000 consecutive elliptic curves, and calculated the excess of positive over negative. We did this many times, and created a histogram plot. The fluctuations look Gaussian! The third appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor N_E, one needs about $\sqrt{N_E} \log N_E$ Fourier coefficients. The fourth appendix gives some estimates on bounding the number of curves in a family with given rank.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to* **C** *and satisfies $-\mathrm{ord}_{s=2} L_2(\mathcal{E}, s)$ $=$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Equidistribution of Signs

ABC Conjecture *Fix $\epsilon > 0$. For co-prime positive integers a, b and c with $c = a + b$ and $N(a,b,c) = \prod_{p|abc} p$, $c \ll_\epsilon N(a,b,c)^{1+\epsilon}$.*

The full strength of ABC is never needed; rather, we need a consequence of ABC, the Square-Free Sieve (see [Gr]):

Square-Free Sieve Conjecture *Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $f(t)$ divisible by p^2 for some $p > \log N$ is $o(N)$.*

For irreducible polynomials of degree at most 3, the above is known, complete with a better error than $o(N)$ ([Ho], chapter 4).

Restricted Sign Conjecture (for the Family $\mathcal{F}$) *Consider a one-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves E_t are equidistributed for $t \in [N, 2N]$.*

The Restricted Sign conjecture often fails. First, there are families with constant $j(E_t)$ where all curves have the same sign. Helfgott [He] has recently related the Restricted Sign conjecture to the Square-Free Sieve conjecture and standard conjectures on sums of Moebius:

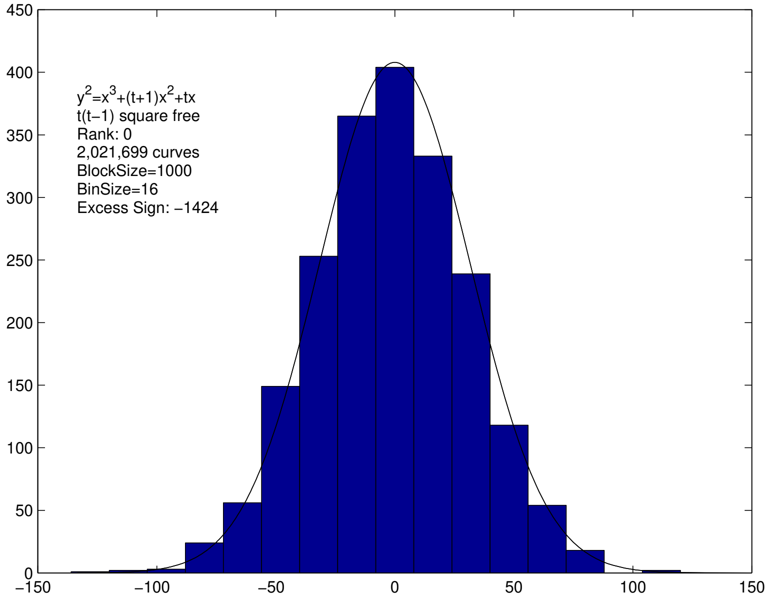
Polynomial Moebius *Let $f(t)$ be a non-constant polynomial such that no fixed square divides $f(t)$ for all t. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.*

The Polynomial Moebius conjecture is known for linear $f(t)$.

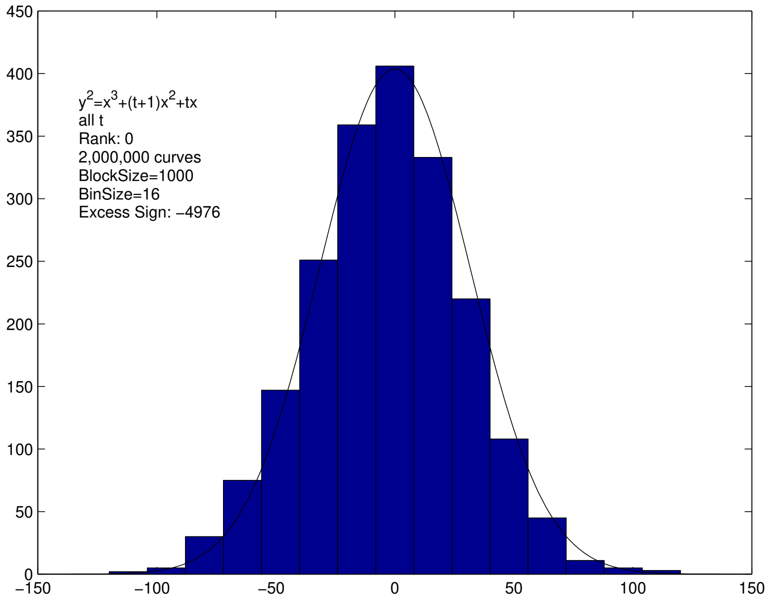
Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

Theorem: Equidistribution of Sign in a Family [He]: *Let $\mathcal{F}$ be a one-parameter family with $a_i(t) \in \mathbb{Z}[t]$. If $j(E_t)$ and $M(t)$ are non-constant, then the signs of E_t, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good t, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.*

Distribution of Signs: $y^2 = x^3 + (t+1)x^2 + tx$

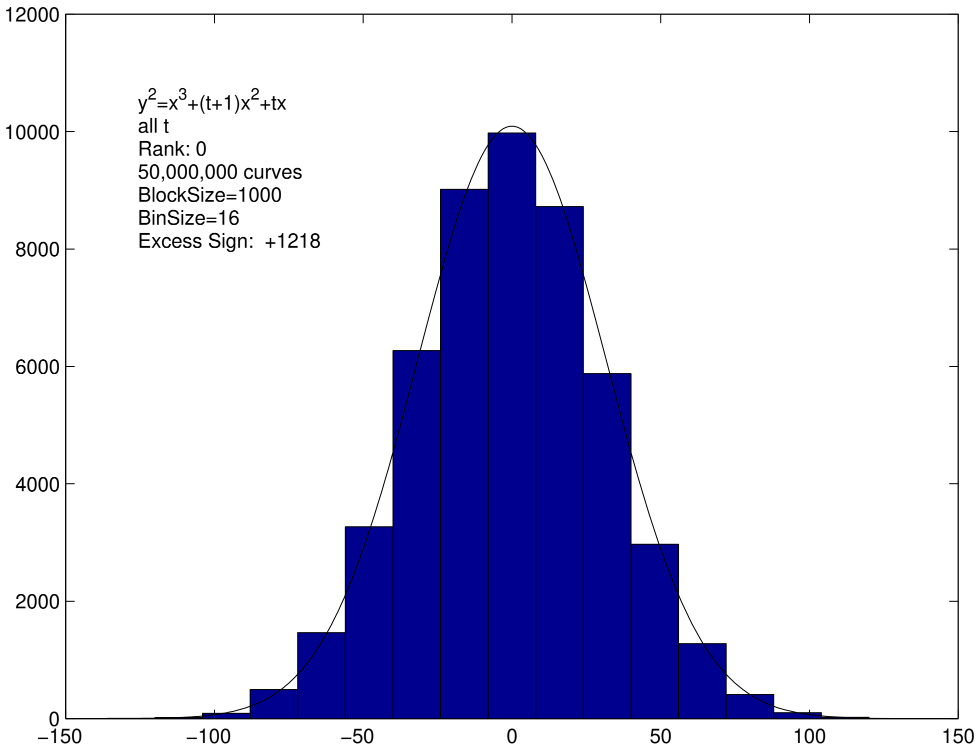


Histogram plot: $D(t)$ sq-free, first $2 \cdot 10^6$ such t.



Histogram plot: All $t \in [2, 2 \cdot 10^6]$.

Distribution of signs: $y^2 = x^3 + (t+1)x^2 + tx$



Histogram plot: All $t \in [2, 5 \cdot 10^7]$

The observed behavior agrees with the predicted behavior. Note as the number of curves increase (comparing the plot of $5 \cdot 10^7$ points to $2 \cdot 10^6$ points), the fit to the Gaussian improves.

Graphs by Atul Pokharel

Harald Helfgott and I can prove that, in some sense, the limiting distribution is the Gaussian. Details will be provided in the near future.

Appendix III: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}
\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.
\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k)\int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy\\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi ny/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix IV: Bounding Excess Rank

$$D_{1,\mathcal{F}}(\phi_1) = \widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0) + r\phi_1(0).$$

To estimate the percent with rank at least $r + R$, P_R, we get

$$R\phi_1(0)P_R \leq \widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0), \ \ R > 1.$$

Note the family rank r has been cancelled from both sides.

The 2-level density gives ***squares*** of the rank on the left, get a cross term rR.

The disadvantage is our support is smaller.

Once R is large, the 2-level density yields better results. We now give more details.

n-Level Density and Excess Rank Bounds

For $n = 1$ and 2, consider the test functions

$$\widehat{f_i}(u) = \frac{1}{2}\left(\frac{1}{2}\sigma_n - \frac{1}{2}|u|\right), \ |u| \leq \sigma$$
$$f_i(x) = \frac{\sin^2(2\pi\frac{1}{2}\sigma_n x)}{(2\pi x)^2}.$$

Expect $\sigma_2 = \frac{\sigma_1}{2}$; only able to prove for $\sigma_2 = \frac{\sigma_1}{4}$.

Note $f_i(0) = \frac{\sigma_n^2}{4}$, $\widehat{f_i}(0) = f_i(0)\frac{1}{\sigma_n}$.

Assume B-SD, Equidistribution of Sign

Notation

Family with rank r, $D_{1,\mathcal{F}}(f) = \widehat{f}(0) + \frac{1}{2}f(0) + rf(0)$.

By even (odd) we mean a curve whose rank r_E has $r_E - r$ even (odd).

P_0: probability even curve has rank $\geq r + 2a_0$.

P_1: probability odd curve has rank $\geq r + 1 + 2b_0$.

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\gamma_E} f\left(\frac{\log N_E}{2\pi}\gamma_E\right),$$

γ_E is the imaginary part of the zeros.

Average Rank: 1-Level Bounds

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E f(0) \ \le\ \widehat{f_1}(0) + \frac{1}{2}f_1(0) + r f_1(0)$$
$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E \ \le\ \frac{1}{\sigma_1} + \frac{1}{2} + r.$$

- All Curves: $r = 0$, $\sigma = \frac{4}{7}$, giving 2.25 (Brumer, Heath-Brown: [Br], [BHB3], [BHB5])
- 1-Parameter Families: $\left(\deg(N(t)) + r + \frac{1}{2}\right)\cdot(1 + o(1))$ (Silverman [Si3]).

Hope 1-Level Density true for $\sigma \to \infty$.

Would yield average rank is $r + \frac{1}{2}$.

Excess Rank: 1-Level Bounds

Assume half even, half odd.

Even curves: $1 - P_0$ have rank $\leq r + 2a_0 - 2$; replace ranks with r. P_0 have rank $\geq r + 2a_0$; replace with $r + 2a_0$.

Odd curves: $1 - P_1$ contributing $r + 1$. P_1 contributing $r + 1 + 2b_0$.

$$\begin{aligned}\frac{1}{\sigma_1} + \frac{1}{2} + r \ &\geq \ \frac{1}{2}\Big[(1 - P_0)r + P_0(r + 2a_0)\Big] \\ &\quad + \frac{1}{2}\Big[(1 - P_1)(r + 1) + P_1(r + 1 + 2b_0)\Big] \\ \frac{1}{\sigma_1} \ &\geq \ a_0 P_0 + b_0 P_1.\end{aligned}$$

1-Level Density Bounds for Excess Rank

$$\begin{aligned} P_0 \ &\leq \ \frac{1}{a_0 \sigma_1} \\ P_1 \ &\leq \ \frac{1}{b_0 \sigma_1} \\ \text{Prob}\{\text{rank} \geq r + 2a_0\} \ &\leq \ \frac{1}{a_0 \sigma_1}.\end{aligned}$$

2-Level Bounds:

$$\begin{aligned} D_{2,\mathcal{F}}(f) &= D^*_{2,\mathcal{F}}(f) - 2D_{1,\mathcal{F}}(f_1 f_2) + f_1(0)f_2(0)N(\mathcal{F},-1) \\ D^*_{2,\mathcal{F}}(f) &= \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du \\ &\quad + r\widehat{f_1}(0)f_2(0) + rf_1(0)\widehat{f_2}(0) + (r^2+r)f_1(0)f_2(0) \\ D_{1,\mathcal{F}}(f) &= \widehat{f}(0) + \frac{1}{2}f(0) + rf(0). \end{aligned}$$

$D^*_{2,\mathcal{F}}(f)$ is over all zeros. Gives

$$\begin{aligned} \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E^2 &\le \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{4} + \frac{1}{3} + \frac{2r}{\sigma_2} + r^2 + r \\ &= \frac{1}{\sigma_2^2} + \frac{2r+1}{\sigma_2} + \frac{1}{12} + r^2 + r + \frac{1}{2}. \end{aligned}$$

Excess Rank: 2-Level Bounds: I

Similar proof yields

Theorem: First 2-Level Density Bounds

$$P_0 \le \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{a_0(a_0+r)}$$

$$P_1 \le \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{b_0(b_0+r+1)}.$$

For $\sigma_2 = \frac{\sigma_1}{4}$, $r = 0$, $a_1 = 1$: ***worse*** than 1-level density.

For fixed $\sigma_2 = \frac{\sigma_1}{4}$ and r, as we increase a_0 we eventually do get a better bound.

Proportional to $\frac{1}{(a_0\sigma_1)^2}$ instead of $\frac{1}{a_0\sigma_1}$.

Excess Rank: 2-Level Bounds: II

Use $D_{2,\mathcal{F}}(f)$ instead of $D^*_{2,\mathcal{F}}(f)$.

$r_E \ = \ $ number of zeros of curve E. Sum over $j_1 \neq j_2$.

r_E even, get $r_E(r_E-2)$ (each zero matched with r_E-2 others).

r_E odd: $(r_E-1)(r_E-2)+(r_E-1) = r_E(r_E-2)+1$.

Theorem: Second 2-Level Density Bounds

$$\begin{aligned} P_0 &\le \frac{\frac{1}{2\sigma_2^2}+\frac{1}{24}+\frac{r}{\sigma_2}-\frac{1}{6\sigma_2}}{a_0(a_0+r-1)} \\ P_1 &\le \frac{\frac{1}{2\sigma_2^2}+\frac{1}{24}+\frac{r}{\sigma_2}-\frac{1}{6\sigma_2}}{b_0(b_0+r)}, \end{aligned}$$

where $a_0 \neq 1$ if $r=0$.

$\sigma_2 = \frac{\sigma_1}{4}$ and $r=0$, better for $a_0 > \frac{\sigma_1^2+8\sigma_1+192}{24\sigma_1}$.
$r=1$, better for $a_0 > \frac{\sigma_1^2+80\sigma_1+192}{24\sigma_1}$.

Decay is proportional to $\frac{1}{(a_0\sigma_1)^2}$.
Note the numerator is never negative; at least $\frac{1}{18}$.

Excess Rank: 2-Level Bounds: IIIa

$r_E \;=\; r + z_E.$

$\sum_{j_1}\sum_{j_2} f_1(L\gamma_{E_{j_1}})f_2(L\gamma_{E_{j_2}})$. Let j_1 be one of the r family zeros, varying j_2 gives $f_1(0)D_{1,E}(f_2)$. Interchanging j_1 and j_2 we get a contribution of $D_{1,E}(f_1)f_2(0)$ for each of the r family.

Only double counting when j_1 and j_2 are both a family zero. Subtract off $r^2 f_1(0)f_2(0)$. For the other z_E zeros: already taken into account contribution from j_1 one of the z_E zeros and j_2 one of the r family zeros (and vice-versa).

Thus, for a given curve, a lower bound of the contribution from all pairs (j_1, j_2) is

$$r f_1(0)D_{1,E}(f_2) + r D_{1,E}(f_1)f_2(0) - r^2 f_1(0)f_2(0) + z_E^2.$$

Excess Rank: 2-Level Bounds: IIIb

Summing over all $E \in \mathcal{F}$ and simplifying gives

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} z_E^2 \ \le\ \frac{1}{\sigma_2^2}+\frac{1}{\sigma_2}+\frac{1}{12}+\frac{1}{2}.$$

Similar calculation gives

Theorem: Third 2-Level Density Bounds

$$P_0 \ \le\ \frac{\frac{1}{2\sigma_2^2}+\frac{1}{2\sigma_2}+\frac{1}{24}}{a_0^2}$$

$$P_1 \ \le\ \frac{\frac{1}{2\sigma_2^2}+\frac{1}{2\sigma_2}+\frac{1}{24}}{b_0+b_0^2}$$

$\sigma_2 = \frac{\sigma_1}{4}$: beats 1-level for $a_0 > \frac{\sigma_1^2+48\sigma_1+192}{24\sigma_1}$.

$r \neq 0$: beats first 2-level once $a_0 > \frac{\sigma_1^2+48\sigma_1+192}{96\sigma_1}$.

$r \ge 1$: beats second 2-level once $a_0 > \frac{3(r-1)}{3r-2}\frac{\sigma_1^2+48\sigma_1+192}{96\sigma_1}$.

Heath-Brown & Brumer

Family of all elliptic curves $E_{a,b}$:

$$\mathcal{F}_T \;=\; \{y^2 = x^3 + ax + b; |a| \le T^{\frac{1}{3}}, |b| \le T^{\frac{1}{2}}.$$

From 1-Level Expansion, get

$$r(E_{a,b}) \;\le\; 2 + \frac{\log T}{\log X} - 2\sum_{p \le X} a_P(E_{a,b}) h\left(\frac{\log p}{\log X}\right) + O\left(\frac{1}{\log X}\right).$$

If $r(E_{a,b}) \ge r \ge 3 + 2\frac{\log T}{\log X}$, then $|U(E_{a,b}, X)| \ge \frac{\log T}{2}$.

Led to

$$\#\{E_{a,b} \in \mathcal{F}_T : r(E_{a,b}) \ge r\} \cdot \left(\frac{\log T}{2}\right)^{2k} \;\le\; \sum_{E_{a,b} \in \mathcal{F}} |U(E_{a,b}, X)|^{2k}.$$

Find $X = T^{\frac{1}{10k}}$, $k = \left[\frac{r-3}{20}\right]$. Yields

$$\begin{aligned} \text{Prob}\left(\text{rank}(E_{a,b}) \ge r\right) &\ll (11r)^{-\frac{r}{20}} \\ \text{rank}(E_{a,b}) &\le 17\frac{\log T}{\log\log T}. \end{aligned}$$

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