An Infinite Family of MSTD Sets

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Introduction

Definition

$[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$
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Given a set A, two basic operations:

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$

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$A = \{1,2,5\}$
$A + A = \{2,3,4,6,7,10\}$
$A - A = \{-4,-3,-1,0,1,3,4\}$

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$A + A = \{2, 3, 4, 6, 7, 10\}$
$A - A = \{-4, -3, -1, 0, 1, 3, 4\}$

Which is bigger?

In general which do we expect to be bigger $|A + A|$ or $|A - A|$?

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Subtraction is not commutative

In general $a - b \neq b - a$, so we expect to have more differences.

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Given a set A, we will call More Sum Than Difference or MSTD if $|A + A| > |A - A|$.

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Are there any MSTD sets?

Yes! The "smallest" is $A = \{1,3,4,5,8,12,13,15\}$. We then have $A+A = [2,30] \setminus \{3,22,29\}$ so $|A+A| = 26$, and $A-A = [-14,14] \setminus \{-13,-6,6,13\}$, so $|A-A| = 25$.

History of the Problem

What proportion of sets are MSTD?

Let $p(n)$ be the proportion of sets in $[1, n]$ that are MSTD. A natural question is does $p(n)$ tend to zero? The answer is no. Martin and O'Bryant (2006) showed that $p(n) > 2 \times 10^{-7}$ for $n \geq 15$.

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Our Result

- Most families of MSTDs have density at most about $2^{-n/2}$.
- Ours has density at least $\frac{1}{n^4}$

Why should there be MSTD sets?

Since $a - b \neq b - a$, we have nearly twice as many "potential differences" as sums. The number of formal sums is

$$\binom{n}{2} + n = \frac{n^2 + n}{2}$$

and the number of formal differences is

$$2\binom{n}{2} + 1 = n^2 - n + 1.$$

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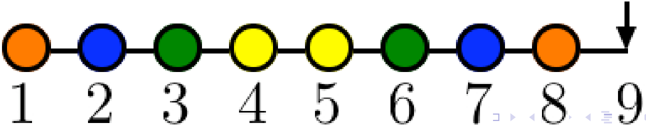


History of the Problem

Expected size of $A+A$ and $A-A$

Let A be a uniformly randomly chosen subset of $[1,n]$ then

$$\Pr(k \in A+A) = \begin{cases} 1-(3/4)^{\frac{k-1}{2}} & 1 \le k \le n \text{ and } k \text{ is odd} \\ 1-(1/2)(3/4)^{\frac{k}{2}-1} & 2 \le k \le n \text{ and } k \text{ is even} \\ 1-(3/4)^{n-\frac{k-1}{2}} & n \le k \le 2n-1 \text{ and } k \text{ is odd} \\ 1-(1/2)(3/4)^{n-\frac{k}{2}} & n < k \le 2n \text{ and } k \text{ is even} \end{cases}$$

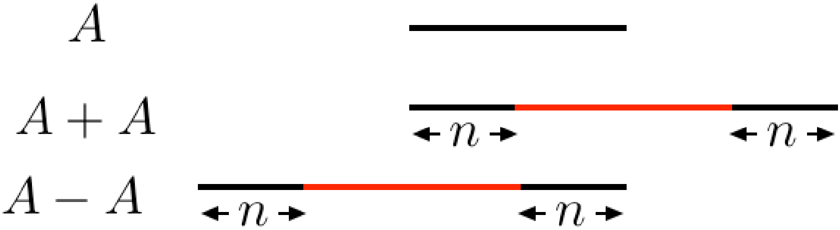
A similar expression holds for $\Pr(k \in A-A)$. It is fairly elementary to sum these and conclude:

$$\mathrm{E}(|A+A|) = (1/2^n) \sum_{A \subset [1,n]} |A+A| = 2n - 11 + o(1)$$

$$\mathrm{E}(|A-A|) = (1/2^n) \sum_{A \subset [1,n]} |A-A| = 2n - 7 + o(1).$$

P_n sets

We say A is a P_n set if both its sum set and its difference set contain all but the n fringe elements on either side.



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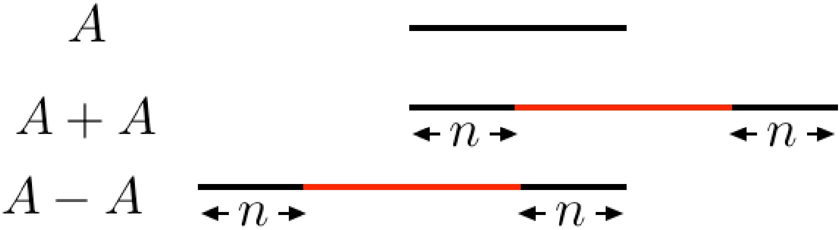
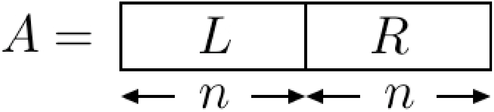


Most sets are P_n

A random set drawn from $[1, 2n]$ in the uniform model is P_n with probability approaching 1.

An Important Lemma

Let $A \subset [1, 2n]$ with $1, 2n \in A$ and suppose that A is P_n. Write $A = L \cup R$ where $L \subset [1, n]$ and $R \subset [n+1, 2n]$.



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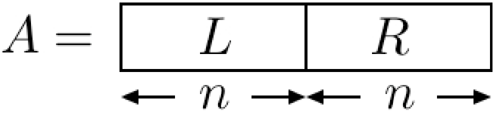
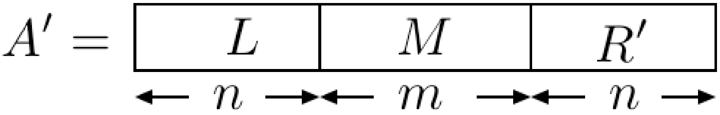


Construct a new set A' by "sticking in" a set M in the middle so that A' is P_n as well. So $A' = L \cup M \cup (R + m)$.



Then A' is P_n, and in general $|A' + A'| - |A + A| = |A' - A'| - |A - A|$.

Constructing new MSTD sets from old

Our objective is to take a P_n, MSTD set $A = L \cup R$ and find a family of sets M so that we can construct a family $A' = L \cup M \cup (R + m)$ of MSTD sets.

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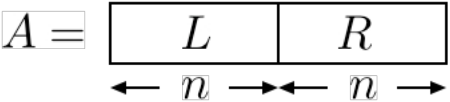
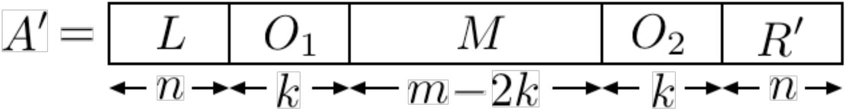
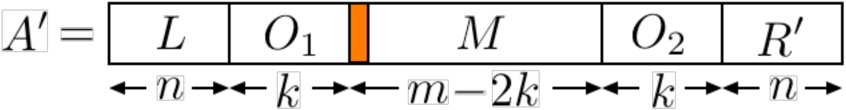


We add three pieces to the middle.

- $O_1 = [n+1, n+K]$ $\leftarrow$ contains k consecutive elements
- $O_2 = [n+m+k+1, n+m+2k]$ $\leftarrow$ k consecutive elements
- M has width $m - 2k$ and no run of k missing elements



How large is this family of sets?



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- We have added elements into a total width of m. So hopefully we have about 2^m possibilities.

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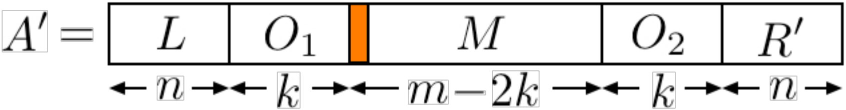

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How many choices for M?

- Divide M into blocks of width $k/2$.
- Assume every block contains at least one element.
- Remove first element for uniqueness

$$\text{Lower bound for fixed k is } 2^{m-2k}\left(1-\frac{1}{2^{k/2}}\right)^{\frac{m-2k}{k/2}}$$

How large is this family of sets?

Number of possible M for a fixed k:

$$2^{m-2k}\left(1-\frac{1}{2^{k/2}}\right)^{\frac{m-2k}{k/2}}$$

Fix the width of the inserted interval, m. The number of possibilities can be found by summing over k:

$$S=\sum_{k=n}^{m/2} 2^{m-2k}\left(1-\frac{1}{2^{k/2}}\right)^{\frac{m-2k}{k/2}}$$

We compute the desired proportion by dividing by 2^m:

$$\frac{S}{2^m}=\sum_{k=n}^{m/2}\frac{1}{2^{2k}}\left(1-\frac{1}{2^{k/2}}\right)^{2m/k}$$

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$$\frac{S}{2^m} = \sum_{k=n}^{m/2} \frac{1}{2^{2k}} \left(1 - \frac{1}{2^{k/2}}\right)^{2m/k}$$

We can achieve bounds:

$$\frac{1}{m^4} < \frac{S}{2^m} < \left(\frac{\log m}{m}\right)^4$$

This is not quite a positive percentage, but much denser than other families of MSTDs.

Questions?