Egg Drop Mathematics:
It IS all its cracked up to be!

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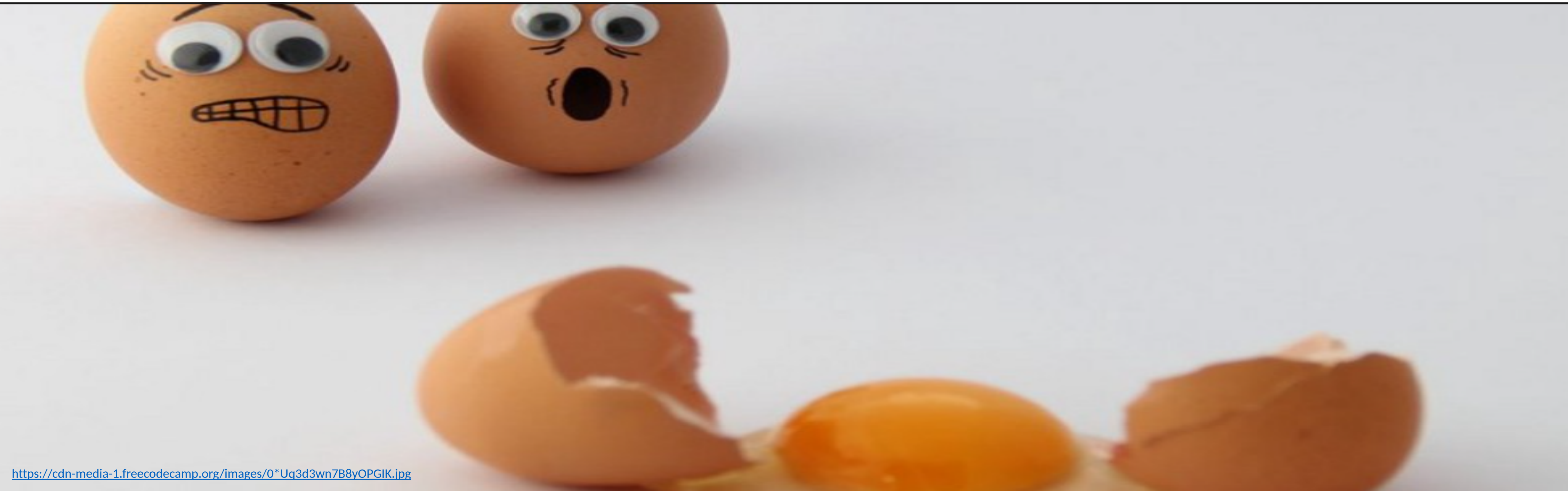

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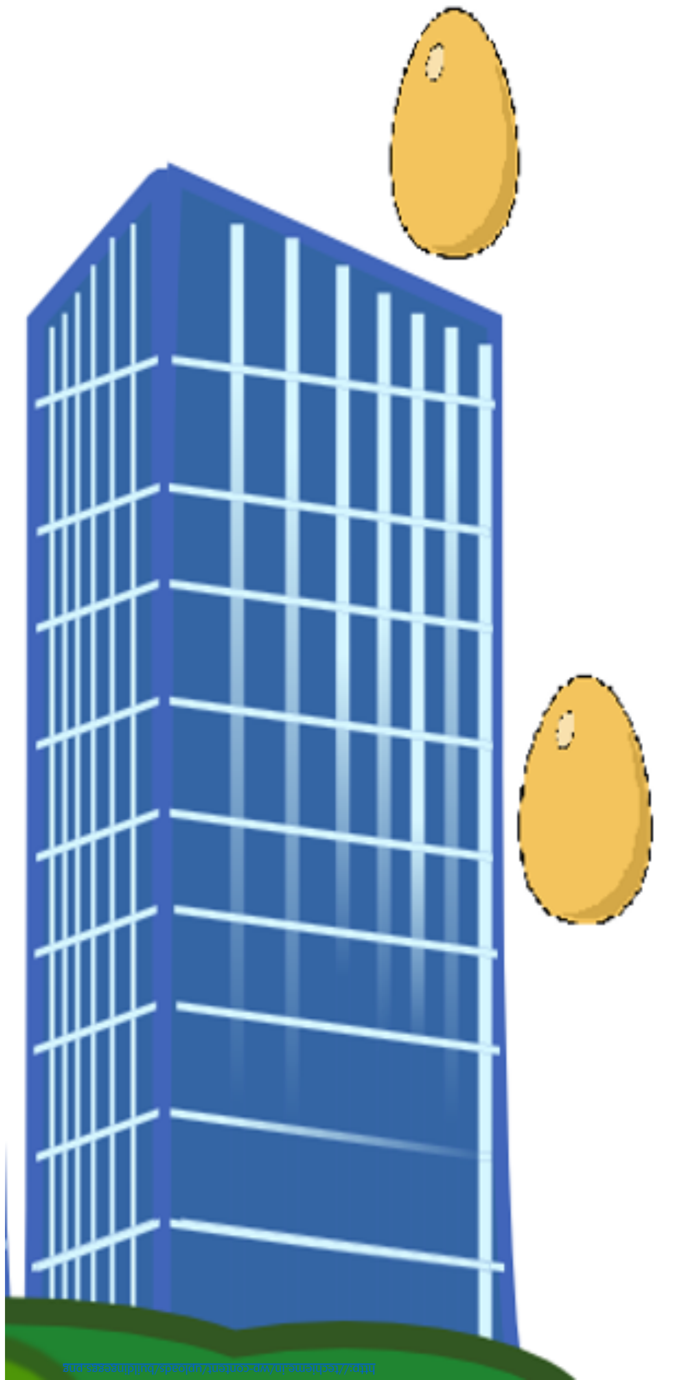
Building with N floors, have 2 golden eggs.

Special eggs: some floor n such that if you drop from below n no damage; can drop as many times as wish.

If drop even once from floor n or higher immediately break.

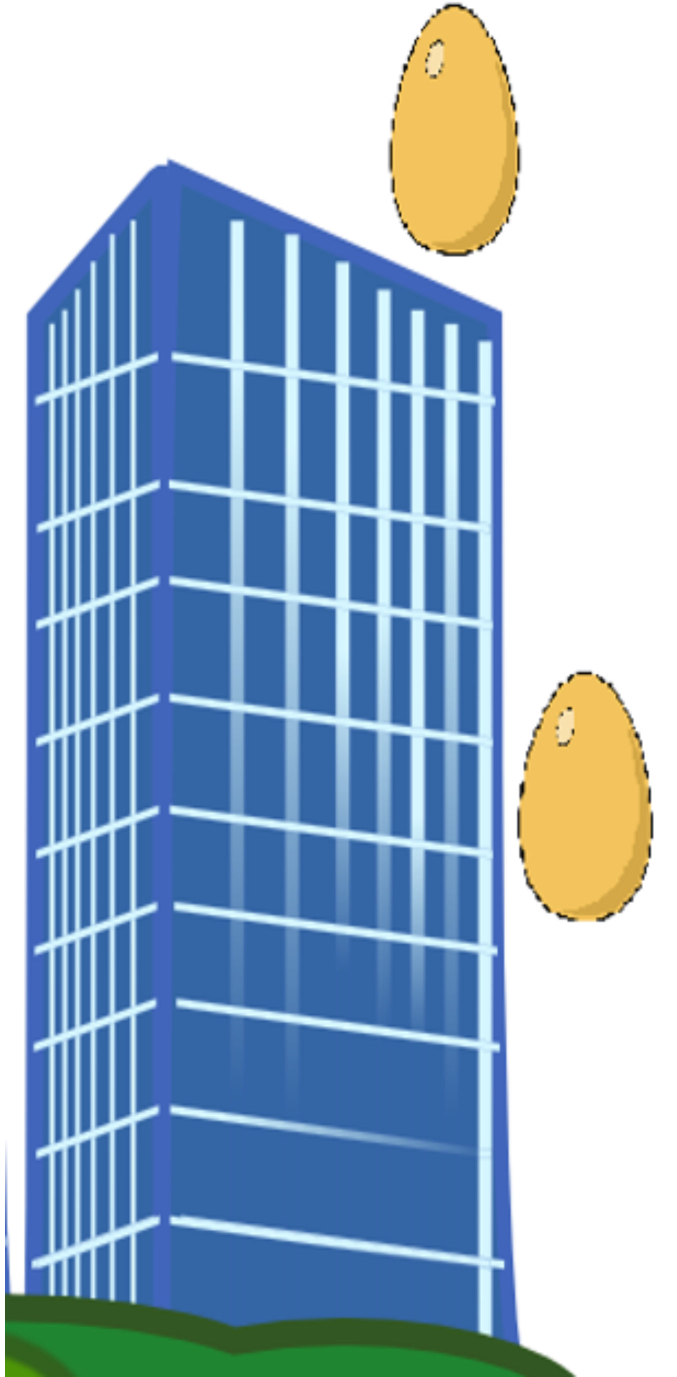
Find in as few drops as you can what n is (the lowest floor where if you drop from there it breaks). Doesn't matter if have any of the golden eggs at the end - just want to know n.



$$\frac{1}{2^{10,000}} = \left(\frac{1}{2^{10}}\right)^{1000} \approx \frac{1}{10^{3000}}$$

Interpretation:

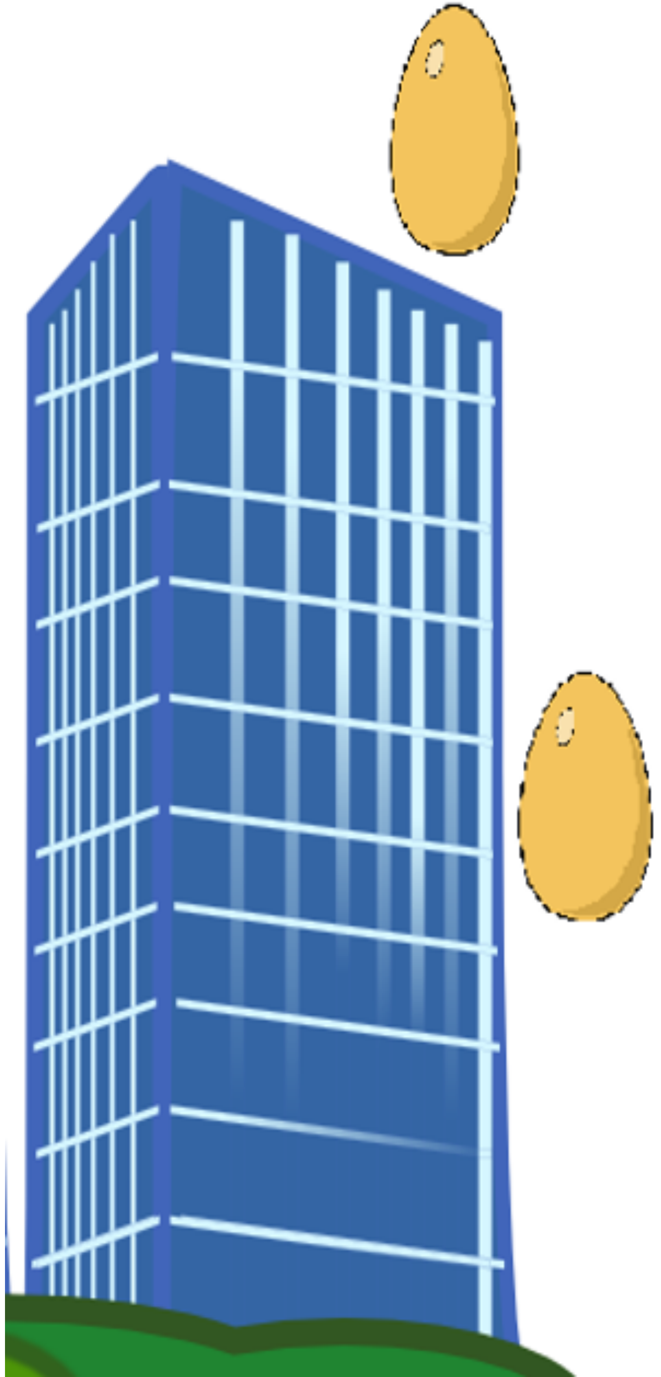
How do you interpret finding n in as few drops as possible?



Interpretation:

How do you interpret finding n in as few drops as possible?

- Minimize worse case.
- Minimize average case.

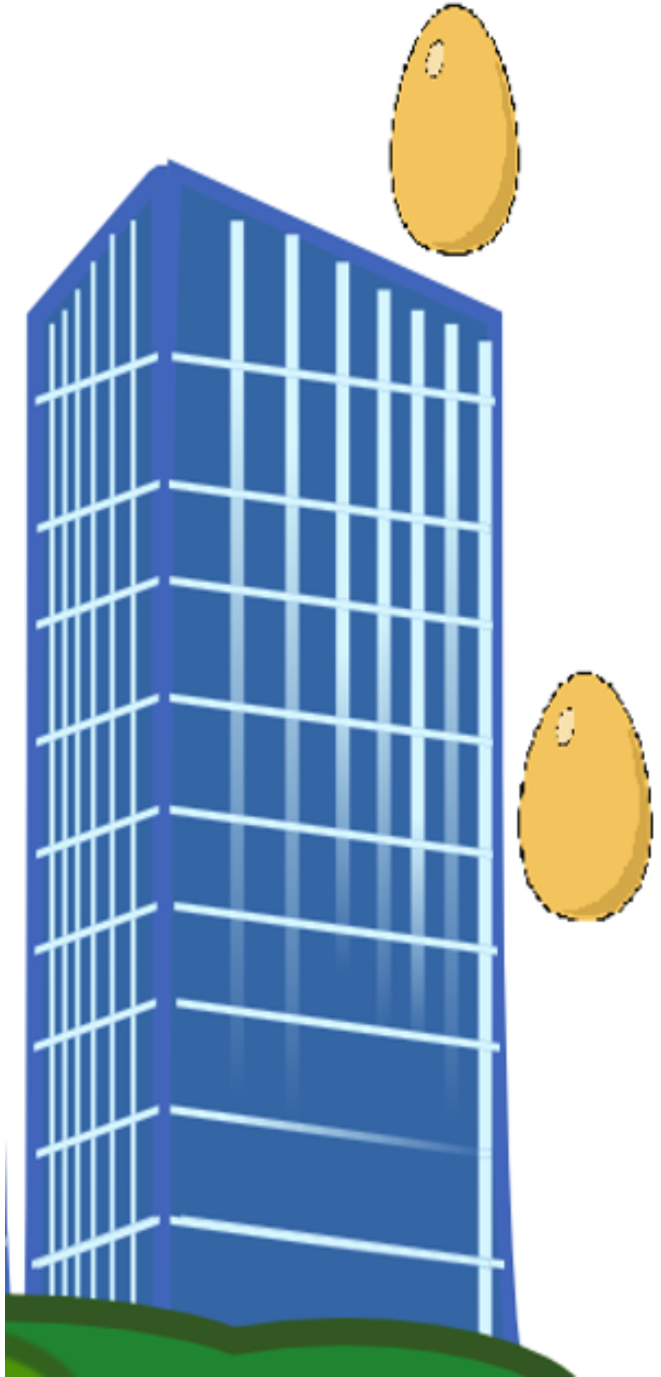


General Advice:

When given a hard problem:

- try to do an easier version first, and
- try to do specific values of parameters.

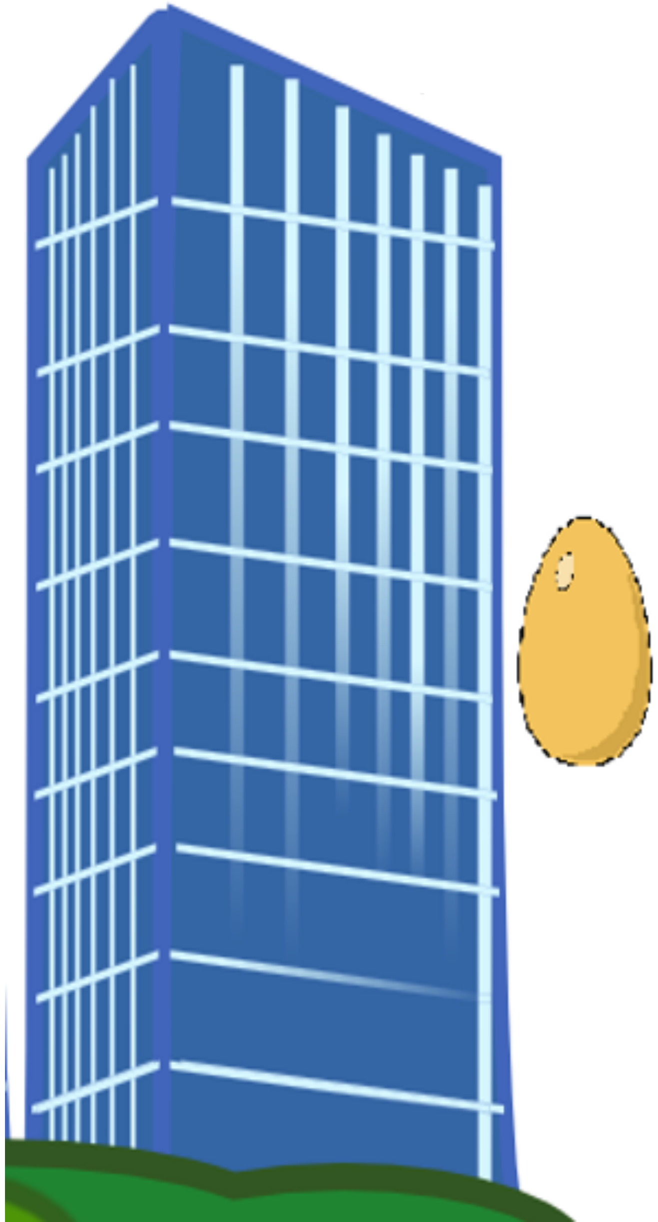
What is an easier problem?



Simple Case: 1 Egg

What is the solution?

1, 2, 3, ..., CRACK



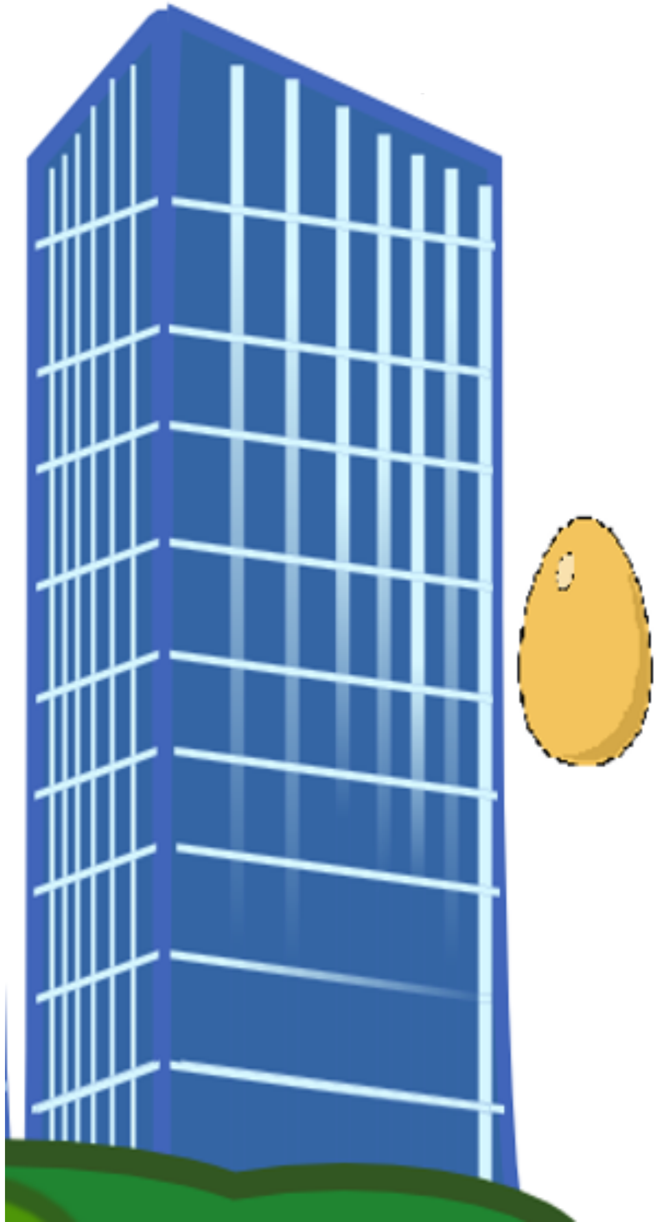
Simple Case: 1 Egg

What is the solution?

Only possibility is go 1, 2, 3, … till break.

Worse case is order N drops.

“1 = −1 = 0”



Next Case: 2 Eggs

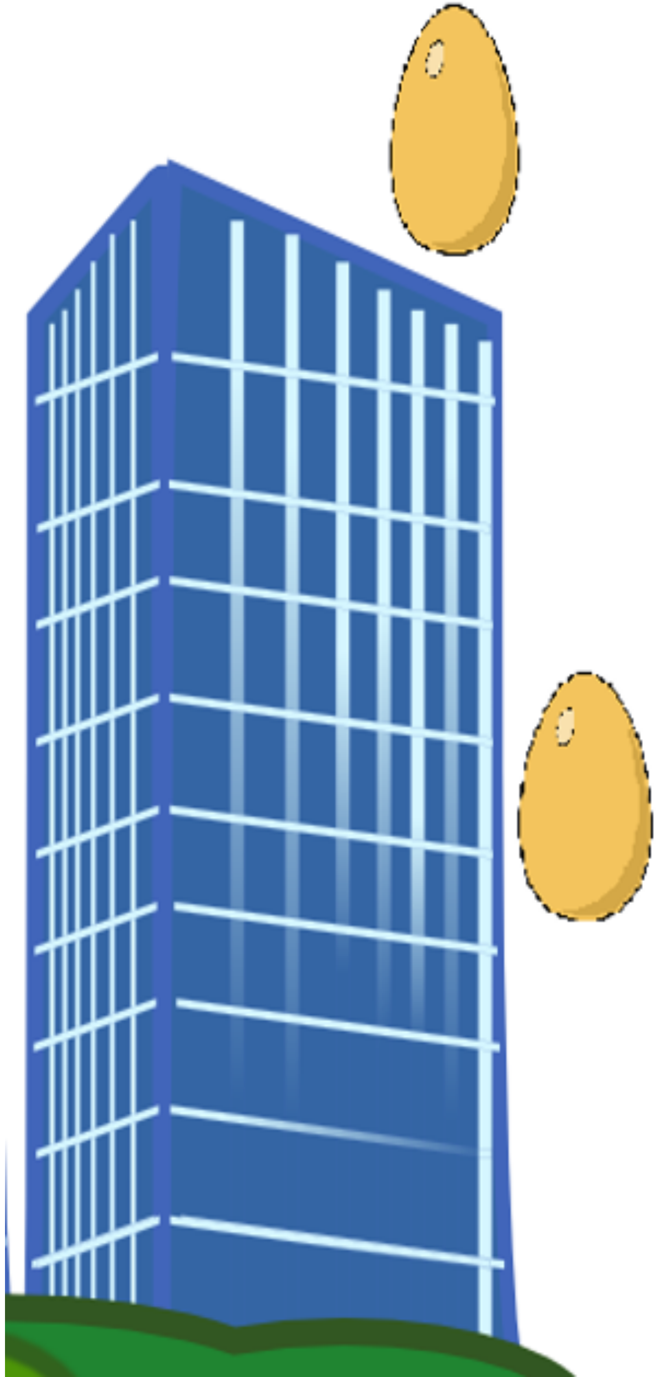
Once one cracks, reduced to 1 egg case.

What are possible strategies?

1) halfway: $N/2$

2) SQUARE ROOT

3) 2, 4, 6, ... $N/2$



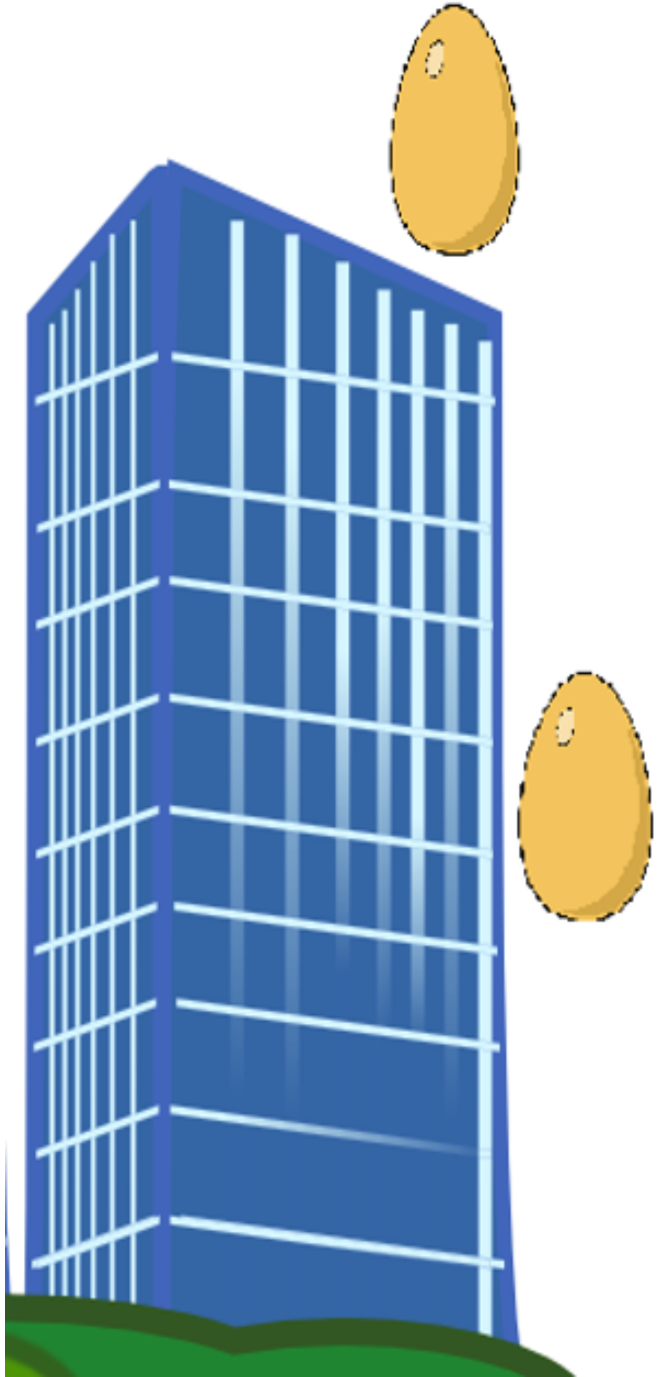
Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?

Extreme cases:

- Drop every 2nd floor.
- Drop at N/2.
- (more generally drop every x)



$x, 2x, 3x, \ldots, \frac{N}{x} x$

$\frac{N}{x} + x - 1$ " $=$ " $\frac{N}{x} + x$

Study $\frac{N}{x} + x$: make small

$\frac{N}{x} = x \Rightarrow x^2 = N \Rightarrow x = \sqrt{N}$

Competing Influences

Drop every 2^{nd} floor.

- Once first breaks fast, but could take many drops.
- #Drops = N/2 + 1

Drop at N/2

- If doesn't crack eliminate a lot, when crack lot to check.
- #Drops = 1 + (N/2 – 1).

Both basically on the order of N/2 drops....

Competing Influences: Balance

Drop every x floors.

Calculus! $\frac{N}{x} + x$

Competing Influences: Balance

Reduced to choosing x to minimize

$$\frac{N}{x} + x \, .$$

Competing Influences: Balance

Reduced to choosing x to minimize

$$\frac{N}{x} + x\,.$$

Set two terms equal to each other to balance:

$$\frac{N}{x} = x \quad so \;\; N = x2 \;\; or \;\; x = N^{1/2}\,.$$

Gives #Drops = $\dfrac{N}{N^{1/2}} + N^{1/2}$ - 1 or about 2 $N^{1/2}$.

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Plot[100/x + x, {x, 1, 100}]
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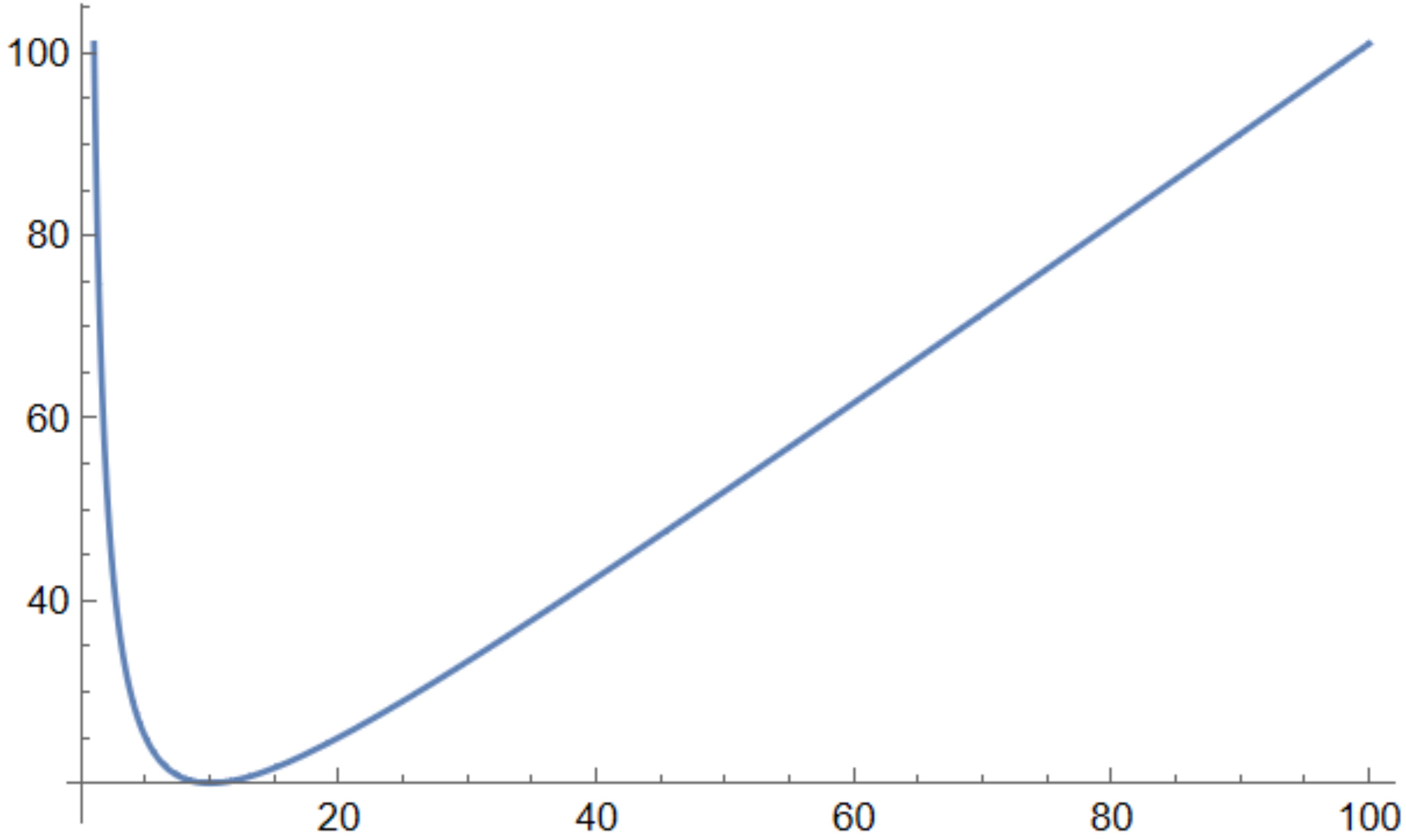
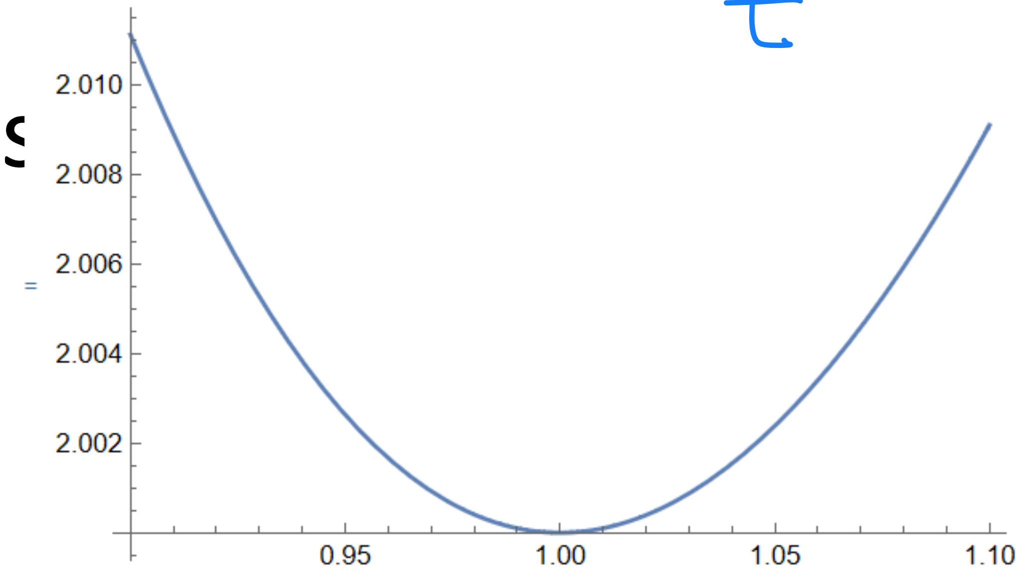


Write x = t $N^{1/2}$ in #Drops = $\dfrac{N}{x} + x - 1$.

Gives #Drops = $\dfrac{N}{t\ N^{1/2}} + t\ N^{1/2}$ - 1.

This is just $N^{1/2}$ $\left(\dfrac{1}{t} + t\right)$,

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Plot[1/t + t, {t, 0, 5}]
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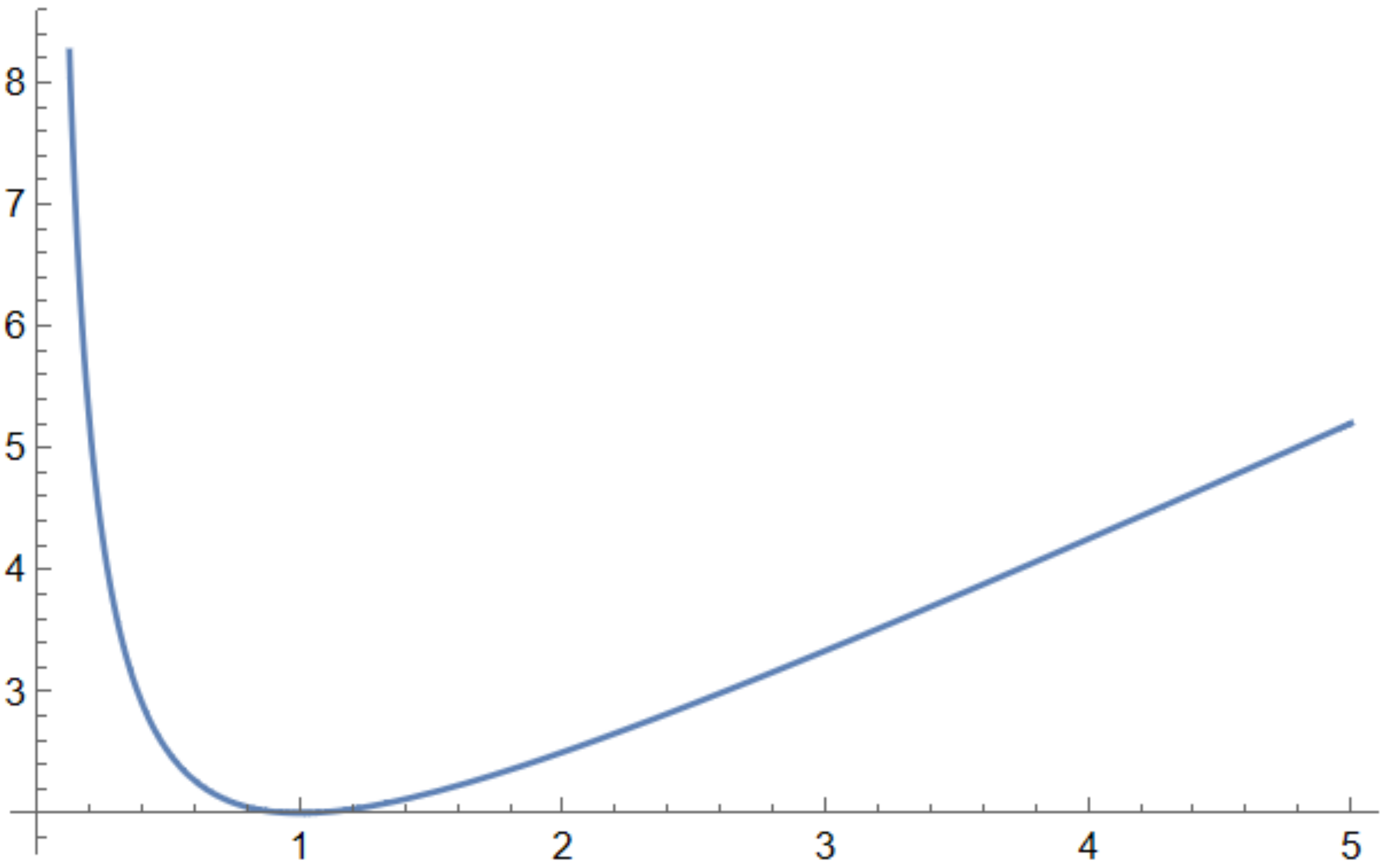

If know calculus: want to minimize $f(x) = N/x + x$:

- Endpoints: $f(1)$ and $f(N)$ are of order N.
- $f'(x) = -N/x^2 + 1$, critical point $f'(x) = 0$ or $x = N^{1/2}$.
- Easily see minimum, or note $f''(x) = 2N/x^3 > 0$.

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds….
…but what if run algorithm 1 and if takes more than 2 seconds on an input switch to first? Average of about 1 second!

Improving Strategy with 2 Eggs

Consider triangular numbers and dynamic rescaline.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
    - Advantage is always same number of drops!
    - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

Improving Strategy with 2 Eggs

Consider triangular numbers and dynamic rescaline.

- Do not move in constant steps of x floors.
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Example: $N = 105 = 14 + 13 + 12 + \ldots + 1$:

$(1 + 13)$ or $(2 + 12)$ or $(3 + 11)$ ….

All are 14 drops, better than $2 * 105^{1/2}$ (about 20).

What if we have 3 Eggs? Or k eggs?

Every - X

$$\frac{N}{X} + \underbrace{2X^{1/2}}_{\text{2 egg problem}}$$

$$\frac{N}{X} = 2X^{1/2}$$

$$N = 2X^{3/2}$$

$$X = N^{2/3} \cdot C_3$$

$$X \sim C_3 N^{2/3}$$

$$\# \text{drops} \sim \tilde{C}_3 N^{1/3}$$

What if we have 3 Eggs? Or k eggs?

For 3 eggs: once one cracks, 2 egg problem.

If do every x it would be, worse case:

# Eggs	Floors for dropping	# drops total
1	$N^{0/1}$	$N^{1/1}$
2	$N^{1/2}$	Const $N^{1/2}$
3	Const $N^{2/3}$	Const $N^{1/3}$
k	Const $N^{\frac{k-1}{k}}$	Const $N^{1/k}$

4 eggs

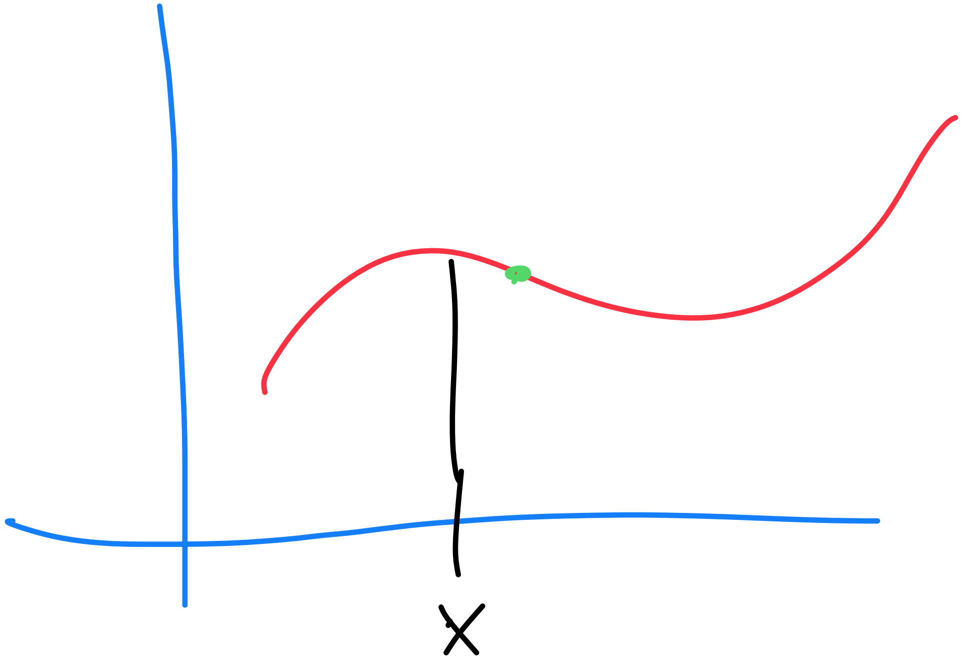
↳ drop at const $N^{3/4}$, till break

and then do 3 egg problem

Digression

$f(x)$

deriv $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x}$



$\frac{\Delta y}{\Delta x}$

$$\left[f_1(x) + f_2(x)\right]' = f_1'(x) + f_2'(x)$$

$$A(x) = f_1(x) + f_2(x)$$

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

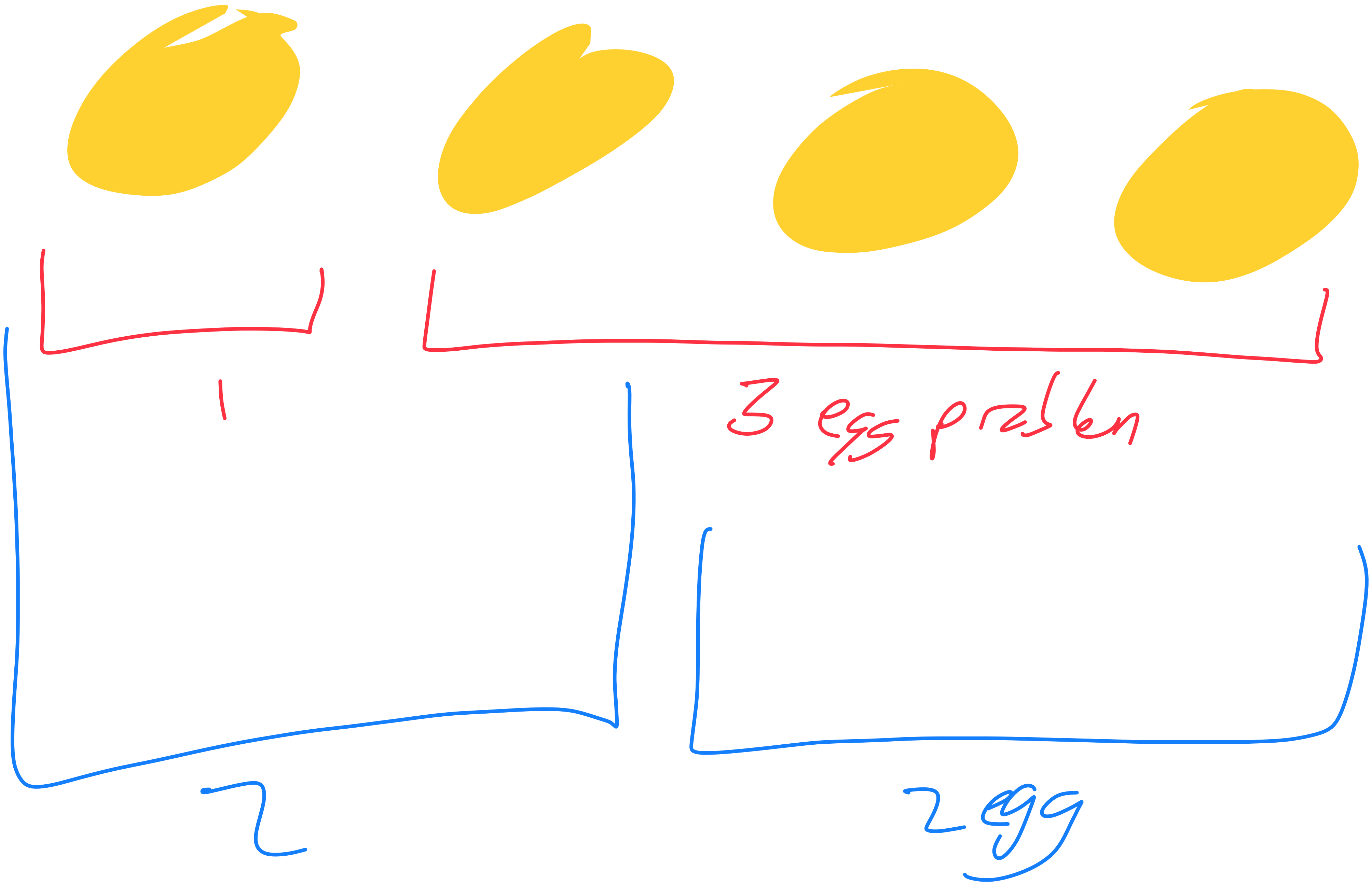
$$\left(\left(f_1(x) + \cdots + f_{n-1}(x)\right) + f_n(x)\right)'$$

$$= \left(f_1(x) + \cdots + f_{n-1}(x)\right)' + f_n'(x)$$

$$\downarrow$$

induction

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(4^n x)}{2^n}$$


1
3 egg problem
2
2 egg

$$\underbrace{1, 2, \ldots, F}_{1}, \underbrace{F+1, \ldots, 2F}_{2}, \ldots$$

N floors,
blocks of F

N/F floors with
problem

2 egg problem for N/F floors

Then 2 egg problem for F floors

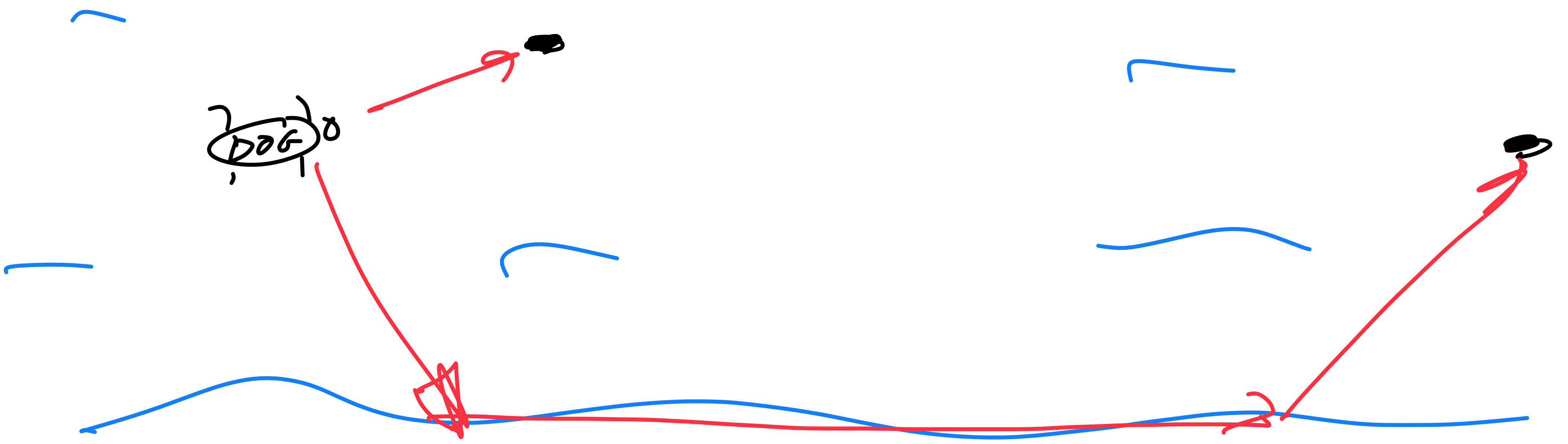
$$\underbrace{2\left(\frac{N}{F}\right)^{\frac{1}{2}}}_{\text{first 2 egg problem}} + 2F^{1/2}$$

Minimize (exclude constants)

$$\frac{N^{1/2}}{F^{1/2}} + F^{1/2} \Rightarrow F = N^{1/2}$$

$$\frac{N^{1/2}}{F^{1/2}} = F^{1/2} \Rightarrow F = N^{1/2}$$

total cost $\quad \dfrac{N^{1/2}}{N^{1/4}} + N^{1/4} = N^{1/4}$



Tim Penning

Elvis

Do dogs know calc

Do dogs know bifurcations