Egg Drop Mathematics:
It IS all its cracked up to be!

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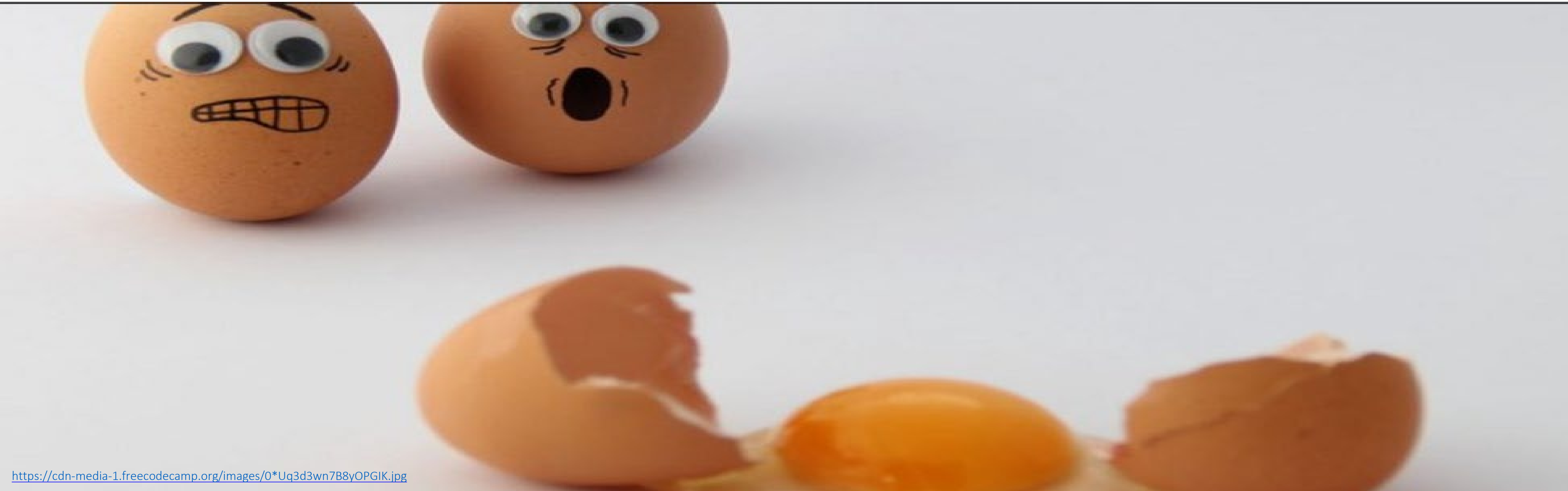
https://web.williams.edu/Mathematics/sjmiller/public_html/



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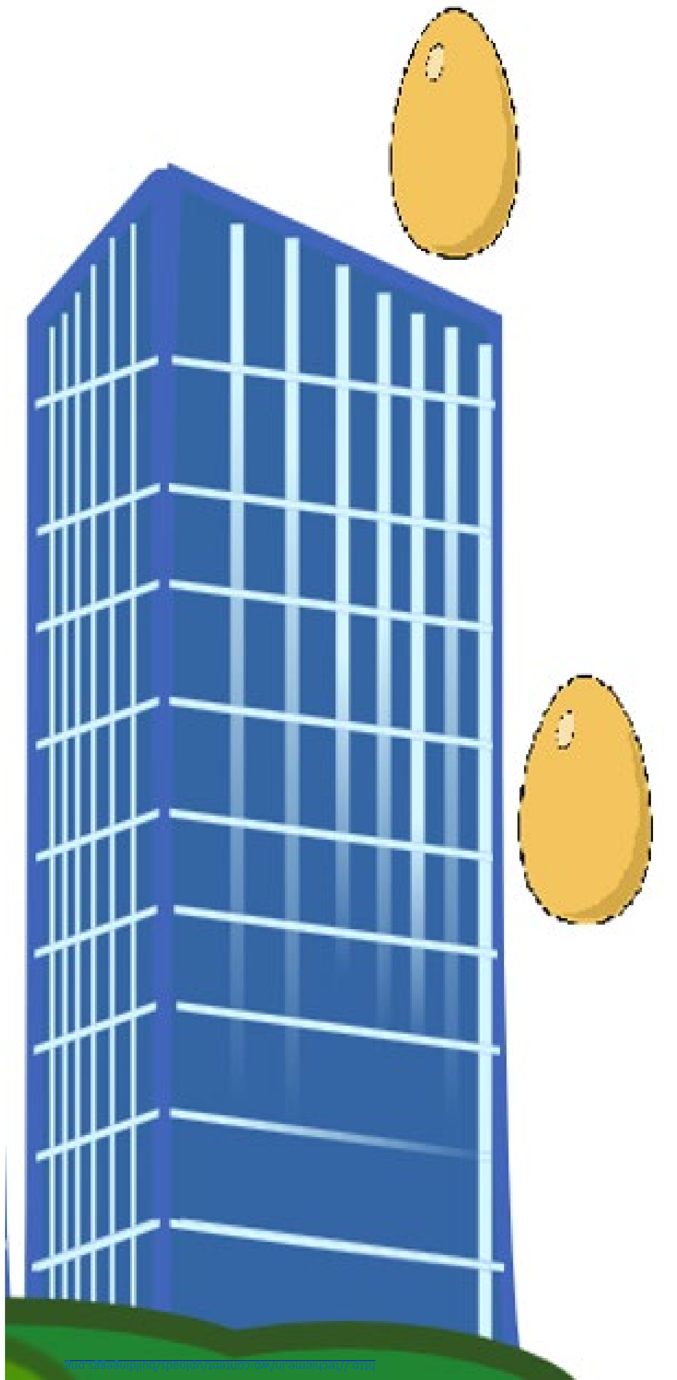
Building with N floors, have 2 golden eggs.

Special eggs: some floor n such that if you drop from below n no damage; can drop as many times as wish.

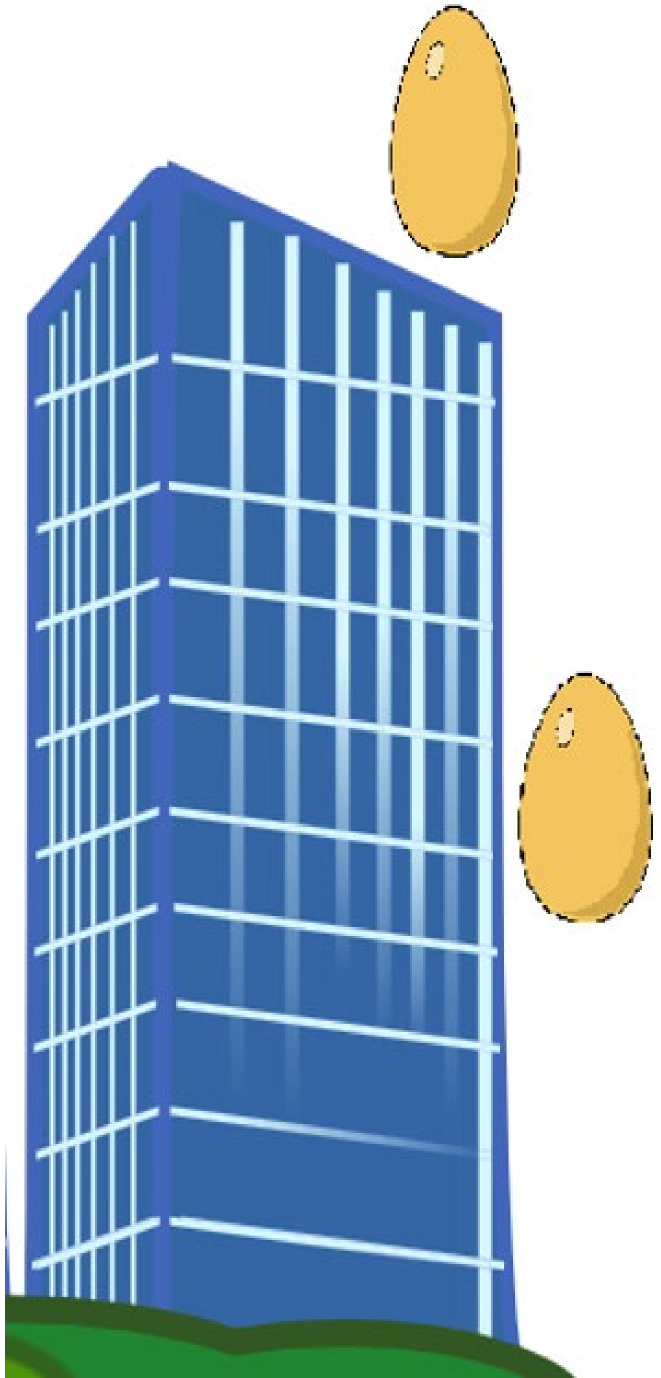
If drop even once from floor n or higher immediately break.

Find in as few drops as you can what n is (the lowest floor where if you drop from there it breaks). Doesn't matter if have any of the golden eggs at the end - just want to know n.



Interpretation:

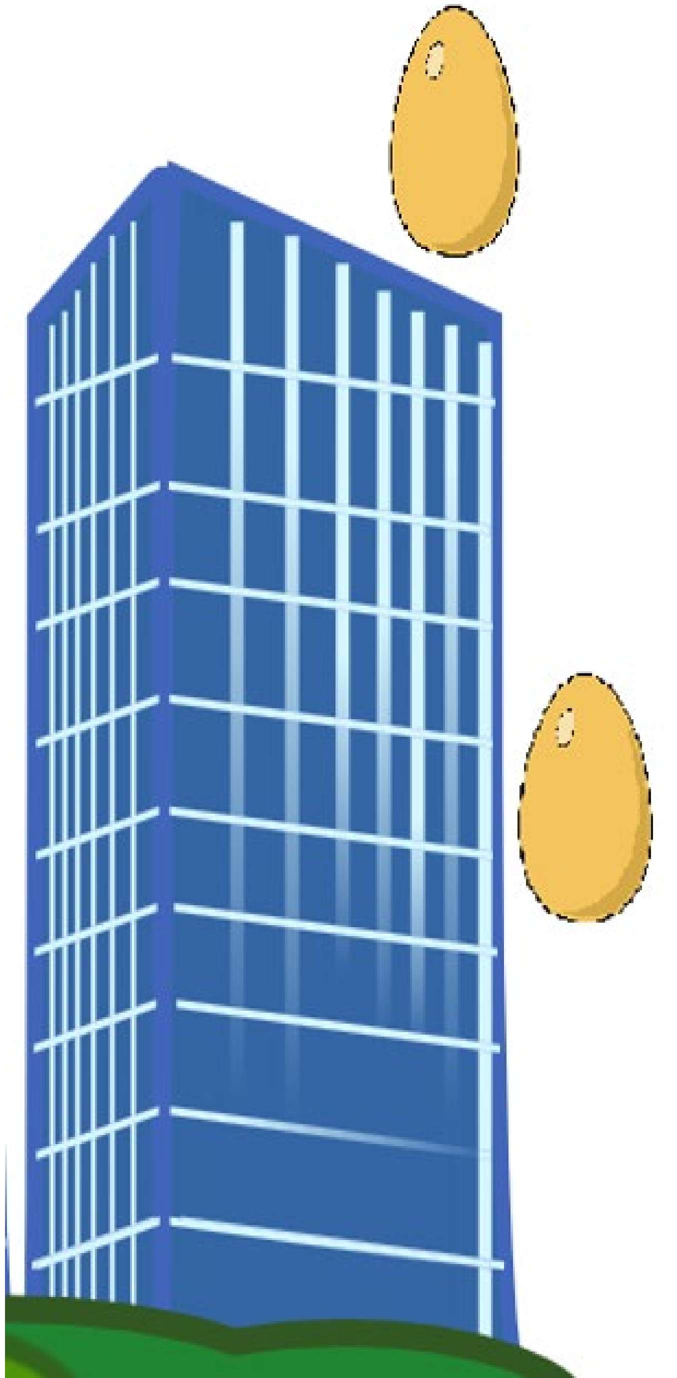
How do you interpret finding n in as few drops as possible?



Interpretation:

How do you interpret finding n in as few drops as possible?

- Minimize worse case.
- Minimize average case.

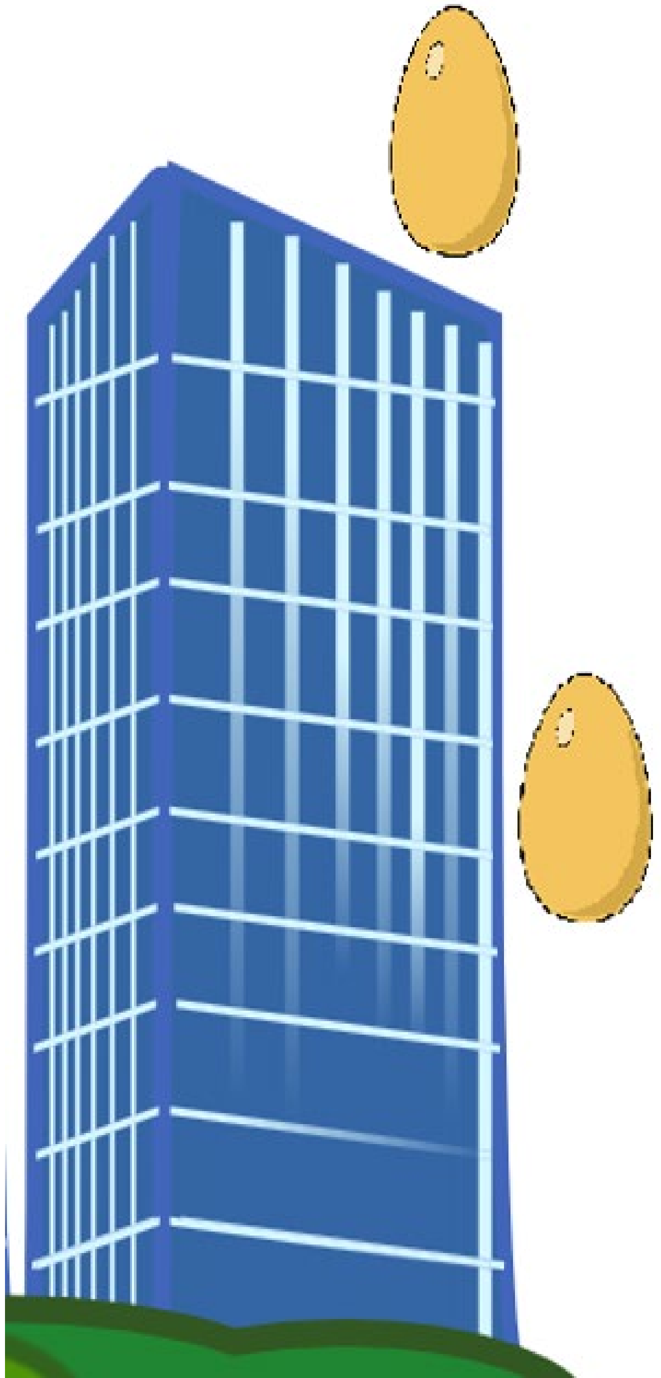


General Advice:

When given a hard problem:

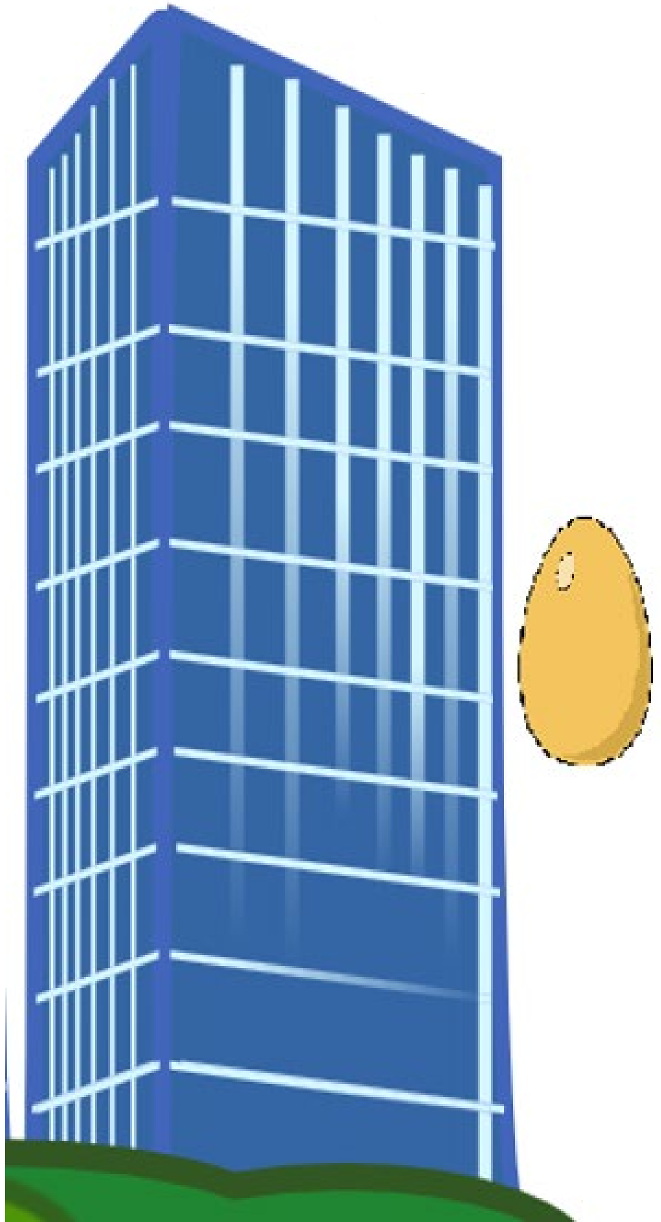
- try to do an easier version first, and
- try to do specific values of parameters.

What is an easier problem?



Simple Case: 1 Egg

What is the solution?

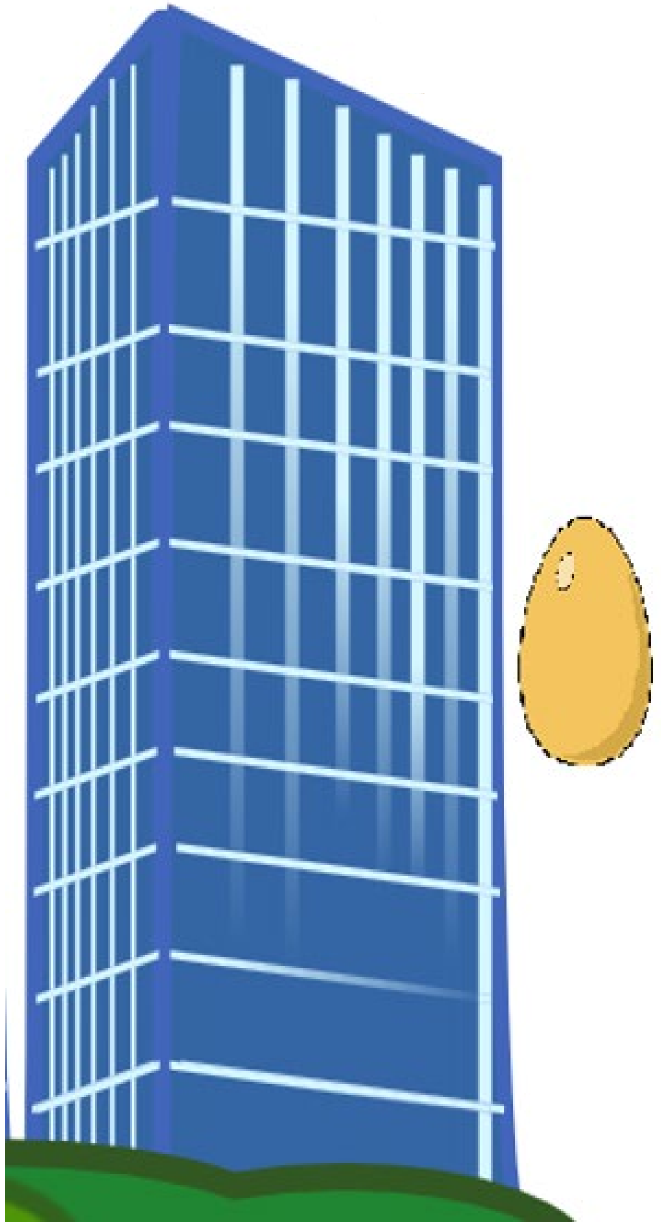


Simple Case: 1 Egg

What is the solution?

Only possibility is go 1, 2, 3, … till break.

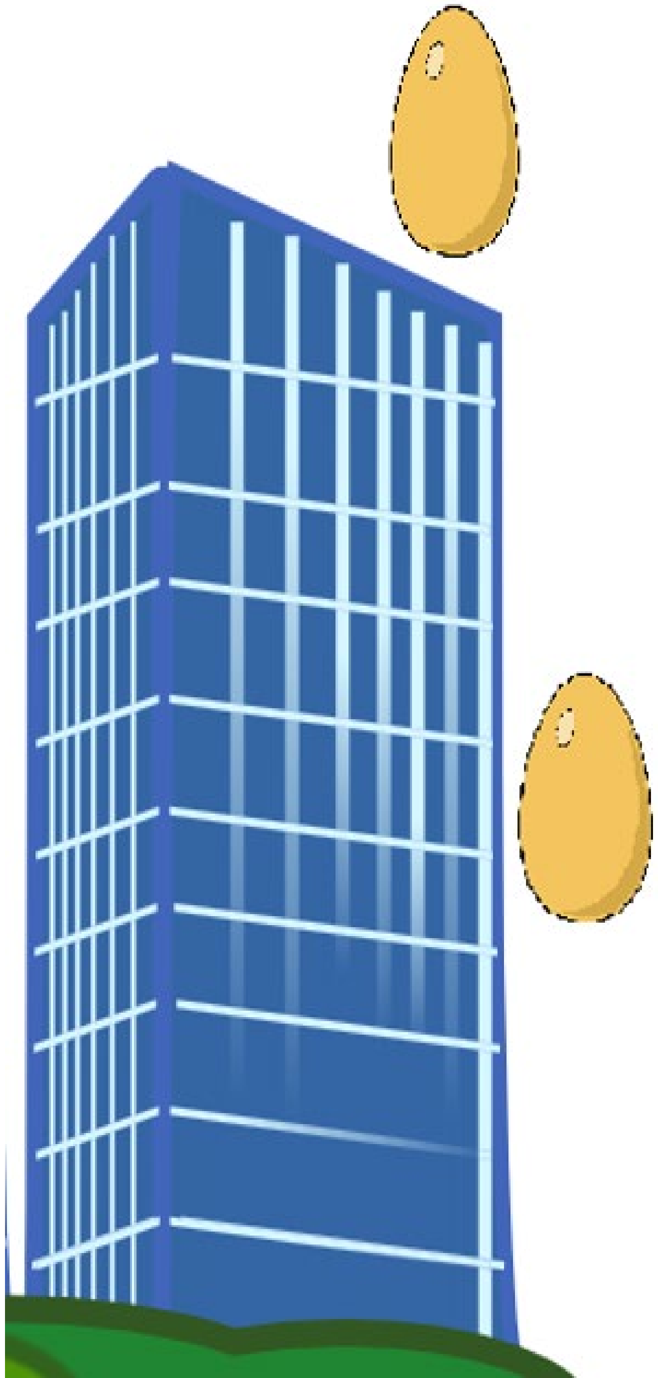
Worse case is order N drops.



Next Case: 2 Eggs

Once one cracks, reduced to 1 egg case.

What are possible strategies?



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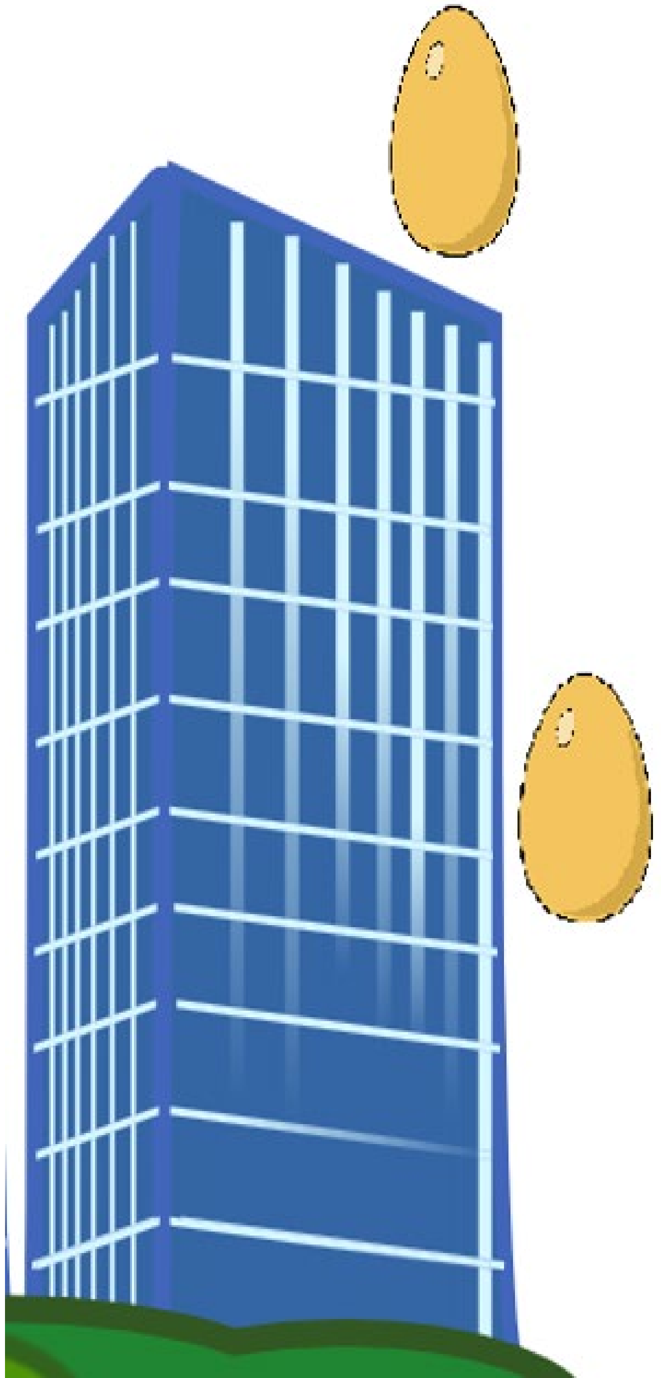

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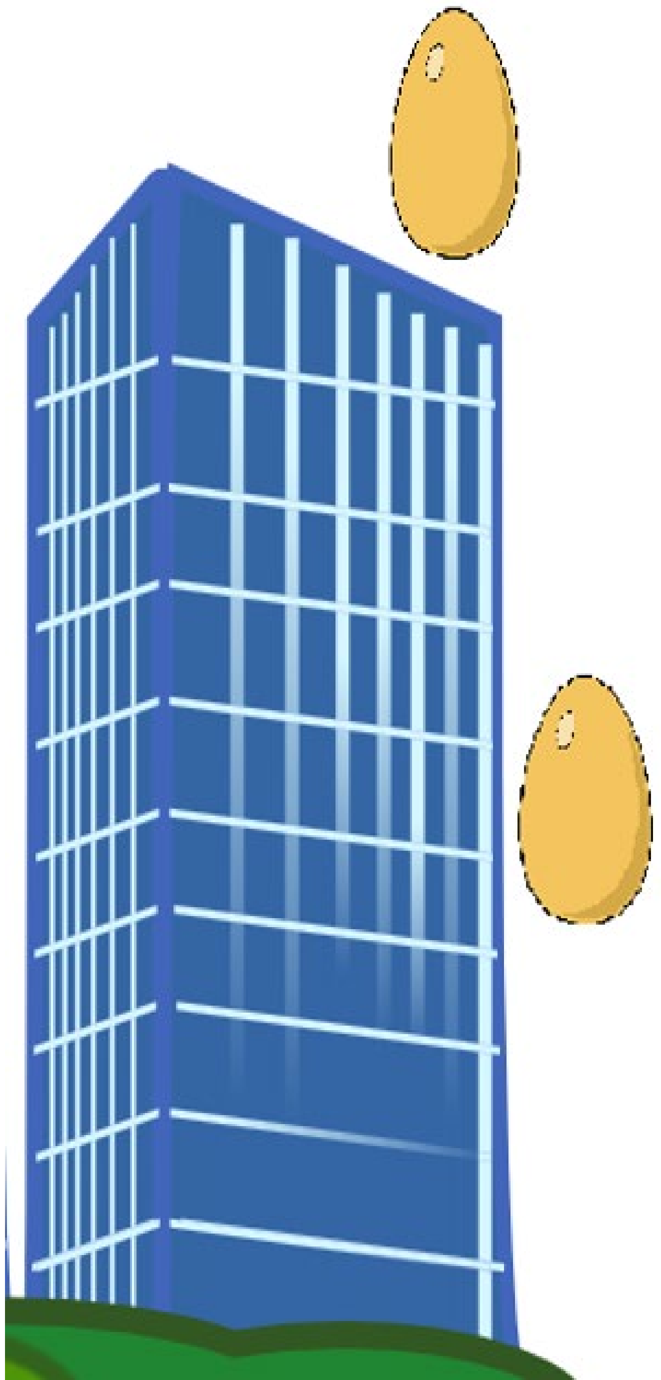
What are possible strategies?

Extreme cases:

- Drop every 2nd floor.
- Drop at N/2.
- (more generally drop every x)



Competing Influences

Drop every 2nd floor.

- Once first breaks fast, but could take many drops.
- #Drops = N/2 + 1

Drop at N/2

- If doesn't crack eliminate a lot, when crack lot to check.
- #Drops = 1 + (N/2 – 1).

Both basically on the order of N/2 drops….

Competing Influences: Balance

Drop every x floors.

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Reduced to choosing x to minimize

$$\frac{N}{x} + x.$$

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Reduced to choosing x to minimize

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Set two terms equal to each other to balance:

$$\frac{N}{x} = x \quad so \quad N = x^2 \quad or \quad x = N^{1/2}.$$

Gives #Drops = $\frac{N}{N^{1/2}} + N^{1/2}$ - 1 or about 2 $N^{1/2}$.

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Plot[100 / x + x, {x, 1, 100}]
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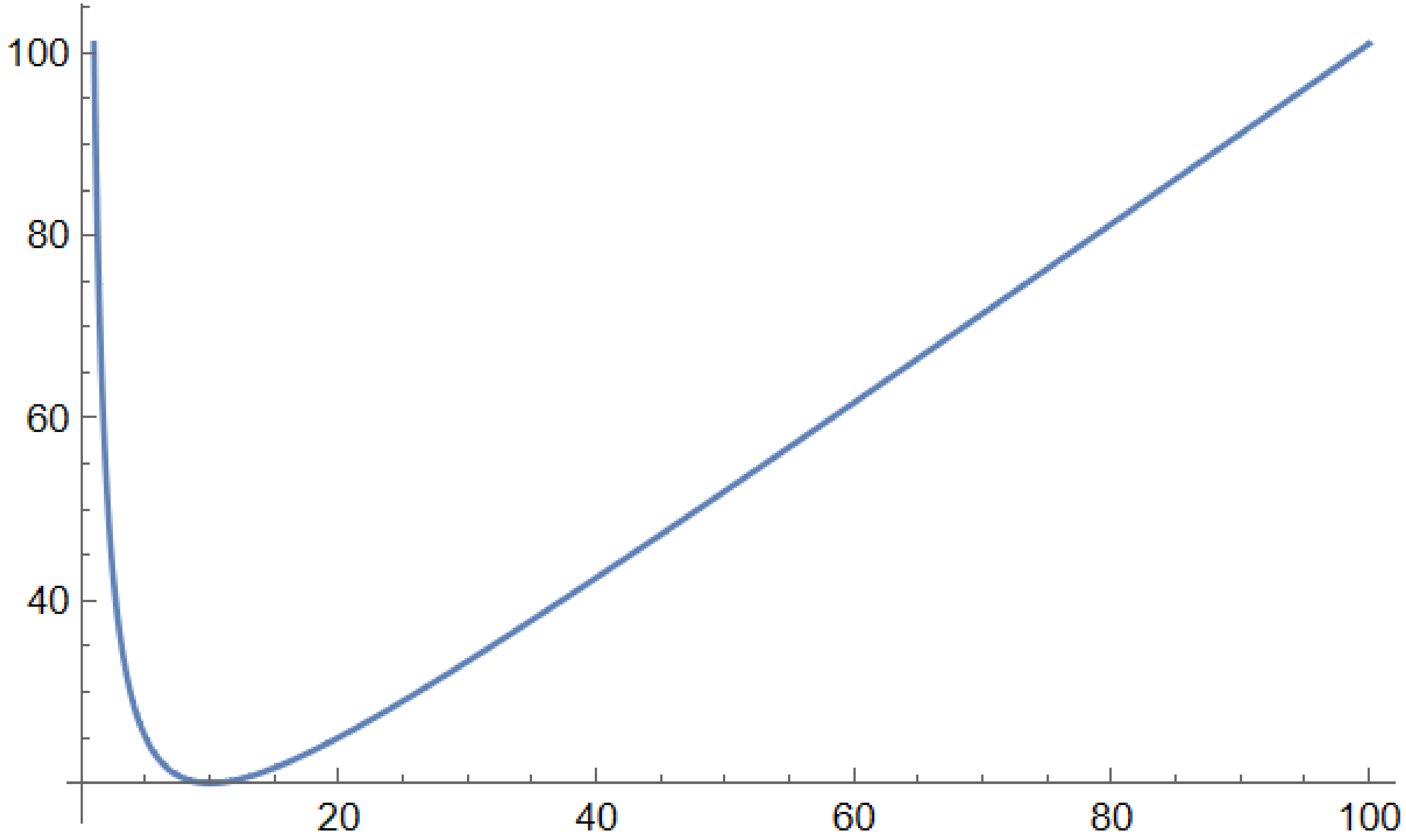
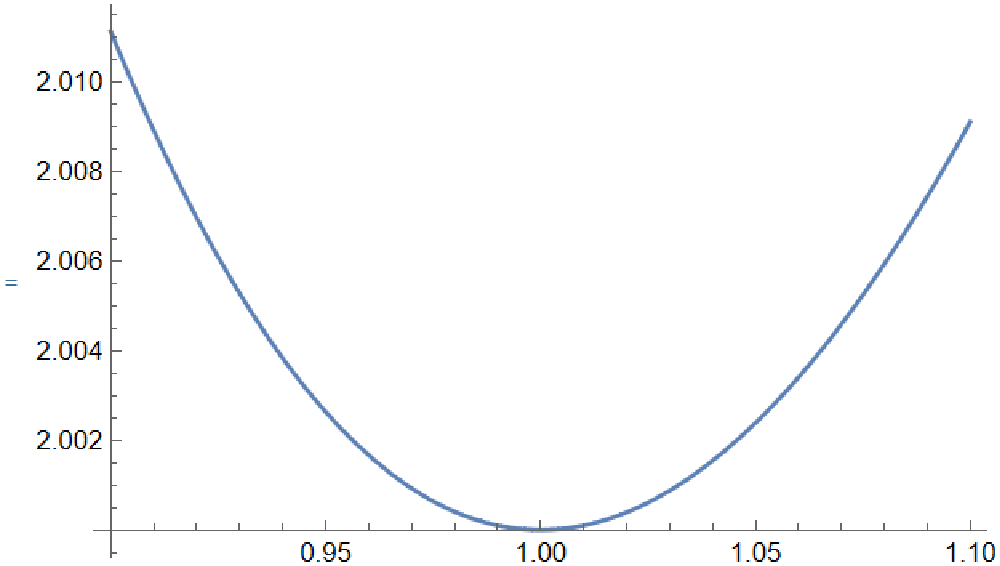


Write x = t $N^{1/2}$ in #Drops = $\frac{N}{x} + x - 1$.

Gives #Drops = $\frac{N}{t\,N^{1/2}} + t\,N^{1/2}$ - 1.

This is just $N^{1/2}$ $(\frac{1}{t} + t)$,

so on the order of $N^{1/2}$!

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Plot[1 / t + t, {t, 0, 5}]
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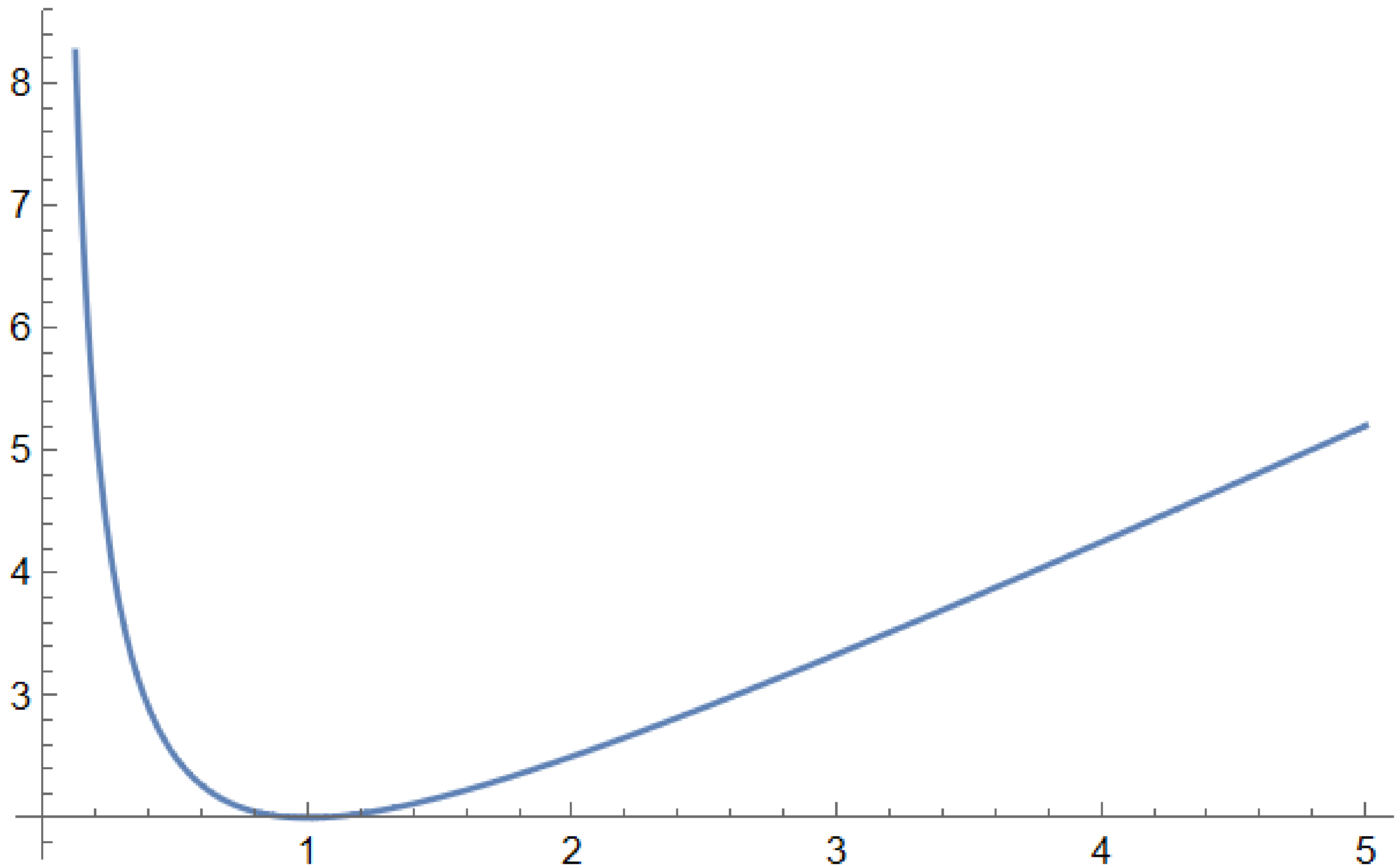

If know calculus: want to minimize $f(x) = N/x + x$:

- Endpoints: $f(1)$ and $f(N)$ are of order N.
- $f'(x) = -N/x^2 + 1$, critical point $f'(x) = 0$ or $x = N^{1/2}$.
- Easily see minimum, or note $f''(x) = 2N/x^3 > 0$.

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
- One takes 1 second except one in a million inputs take 1,000,000,000 seconds.

Both take on average approximately 1000 seconds....

Balancing Application

Imagine have two algorithms:

- One always takes 1000 seconds.
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Both take on average approximately 1000 seconds....
...but what if run algorithm 1 and if takes more than 2 seconds on an input switch to first? Average of about 1 second!

Improving Strategy with 2 Eggs

Consider triangular numbers and dynamic rescaline.

- Do not move in constant steps of x floors.
- Do x, then x-1 if doesn't crack, then x-2….
    - Advantage is always same number of drops!
    - Basically if doesn't crack doing 2 egg problem but now with N-x floors (after first drop).

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Example: N = 105 = 14 + 13 + 12 + … + 1:

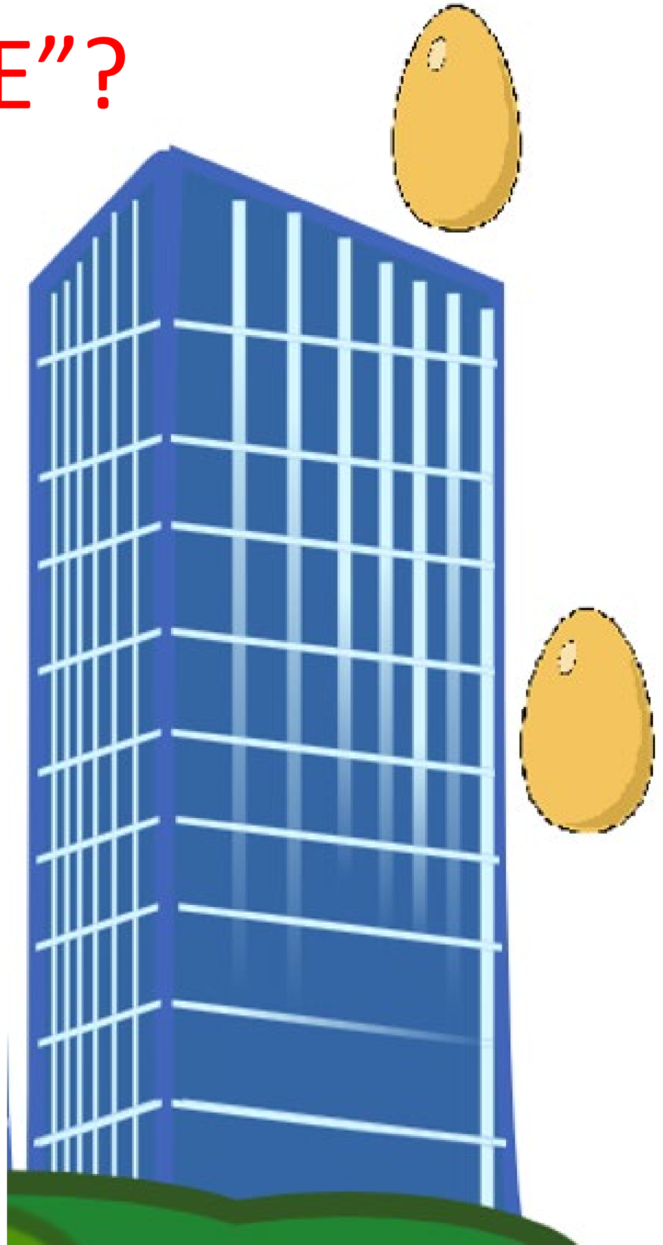
(1 + 13) or (2 + 12) or (3 + 11) ....

All are 14 drops, better than 2 * $105^{1/2}$ (about 20).

What if we have 3 Eggs? Or k eggs?

NEW RESEARCH QUESTION: Email sjm1@williams.edu What if "TWO-DIMENSIONAL"? Or "THREE"?

- Consider box from (0,0) to (M,N), find special point (m,n) such that if drop at (a,b) with a < m and b < n no damage, otherwise breaks.
- What if breaks only when m+n > V?
- What if breaks only when am + bn > V?



What if we have 3 Eggs? Or k eggs?

For 3 eggs: once one cracks, 2 egg problem.
If do every x it would be, worse case: