On Some Observations about a prime factorization sequence called F-palindromes

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$$2 + (3 + 2) + 11 = (3 + 4) + 11 = 18$$

Formal definition of v-palindrome

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Definition 2. *Let* $b \geq 2$*, then* $n = (a_{L-1}a_{L-2}\ldots a_1a_0)_b$ *denotes the number* $\boldsymbol{n}$ ***in base b****. The function* $r_b : \mathbb{N}_{\neq b} \to \mathbb{N}_{\neq b}$*, where* $\mathbb{N}_{\neq b} = \{n \in \mathbb{N} | b \nmid n\}$*, is the* ***reversal function in base b*** *defined to be (using the same* n *as earlier):*

$$r_b(n) := (a_0a_1\ldots a_{L-2}a_{L-1})_b.$$

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$$r_b(n) := (a_0a_1\ldots a_{L-2}a_{L-1})_b.$$

- Definition of v-function:

$$v(p^\alpha) := p + \iota(\alpha),$$

where the function $\iota(\alpha)$ is defined as

$$\iota(\alpha) := \alpha[\alpha > 1],$$

Formal definition of v-palindrome cont.

Definition 3. *A number* $n \in \mathbb{N}_{\neq b}$ *is said to be a* ***v-palindrome in base b*** *if the following hold:*

1. n *is not a palindrome (i.e.* $n \neq r_b(n)$*)*
2. $v(n) = v(r_b(n))$.

Nonexistence of prime v-palindromes in base 10

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$$v(r_{10}(p))) \leq r_{10}(p) < p = v(p)$$

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Lemma 4. *Let $x, y \in \mathbb{N}$. Suppose $D_b(x) = n$ and $D_b(y) = m$. Then:*

$$D_b(xy) = n + m - \left[xy < b^{n+m-1}\right].$$

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Lemma 6. *Let p prime where $D_{10}(p) = n, n \geq 5$ and q_1 is the biggest prime factor of $r(p)$ where r_1 is exponent of q_1. If $D_{10}(p) - D_{10}(q_1) \geq 2$, then $(n-1) \leq D_{10}(p - q_1 - r_1) \leq (n)$.*

Case 2: r(p) > p (cont.)

Lemma 11. *Let p prime where $D_{10}(p) = n$, $n \geq 5$ and q_1 is the biggest prime factor of $r(p)$. If $D_{10}(p) - D_{10}(q_1) = 0$, then $v(r(p)) \neq v(p)$ except possibly primes of the form $(499\cdots999)$ whose middle digits are 9s.*

v-palindromes in Base p+1

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Lemma 13. *Let* $q \in \mathbb{P}$. *Then* $v(2q) = v(q^2)$

v-palindromes in Base p+1

Theorem 1. *If $p > 2$ (that is, p is an odd prime), then $2p, p^2 \in Pal(v, p+1)$ (i.e. $2p$ and p^2 are v-palindromes in base $p+1$).*

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Theorem 2. *If $p > 2$, and $p \nmid k$, then $2p \cdot \rho_{p+1,2}(k), p^2 \cdot \rho_{p+1,2}(k) \in Pal(v, p+1)$*

base 4	**base 6**	**base 8**
12	14	16
1212	1414	1616
121212	141414	161616
12121212	14141414	16161616
…	…	…

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Theorem 3. *If $p > 2$ and $p \nmid k+1$, then $(1\,[p\rho_{1,p+1}(k)]\,[p-1])_{p+1}, ([p-1]\,[p\rho_{1,p+1}(k)]\,1)_{p+1} \in Pal(v, p+1)$*

<u>**base 4**</u>	<u>**base 6**</u>	<u>**base 8**</u>
12	14	16
132	154	156
1332	1554	1556
13332	15554	15556
…	…	…

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Theorem 5. *If* $p > 2$, *then* $2p^2 \cdot \rho_{p^2+1,2}(k), p^2 \cdot \rho_{p^2+1,2}(k) \in Pal(v, p^2 + 1)$

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Theorem 6. *If* $p > 2$, *then* $(1 \left[p^2 \rho_{1,p^2+1}(k)\right] \left[p^2 - 1\right])_{p^2+1}, (\left[p^2 - 1\right] \left[p^2 \rho_{1,p^2+1}(k)\right] 1)_{p^2+1} \in Pal(v, p^2 + 1)$

Generalizations of v-palindromes in Base b

- The idea came about trying to prove that there exists a v palindrome in every base.
- To prove such statement, we had to prove at least one occurrence.
- From this, generalizations of v palindromes were formed.

Now think of $v(n)$ is an additive function so that for a $p, q \in \mathbb{N}$ where $p < t$, for every integer $t \geq 1$, with $(t, pq) = 1$, $v(pt) = v(qt)$.

Generalizations of v-palindromes in Base b

- Then for this truly be arbitrary:

Lemma 15. *For a* $p \neq r_b(q)$, *there exists a* p, q *such that* $v(p) = v(q)$.

Generalizations of v-palindromes in Base b

- Tsai originally provided an example of a possible situation using the fact that v(5) = v(6)
- From this work, we were able to generalize the process such that the following is true:

Theorem 8. *Let $b \geq 2$ be an integer. If there exist distinct integers $p, q \geq 1$ such that*

$$v(x) = v(y),$$
$$b \equiv 4xy \pmod{xy(x+y)},$$

then there exists a v-palindrome in base b.

Generalizations of v-palindromes in Base b

- Looked at more than 2 digit v-palindromes. Needed a way to specify bases with digits:

Now consider a base $b \geq 2$. Let us take an arbitrary number in base b say n_b. Decompose this n_b such that $n_b = (d_n.......d_1)_b = \sum_{i=0}^{n} d_i b^i$. As such, $(r(n))_b = (d_1.......d_n)_b = \sum_{i=0}^{n} d_i b^{n-i}$. By definition then, if $v((d_n....d_1)_b) = v((d_1.......d_n)_b)$, then

$$v\left(\sum_{i=0}^{n} d_i b^i\right) = v\left(\sum_{i=0}^{n} d_i b^{n-i}\right)$$

For some $t \geq 1$, with $(t, pq) = 1$, the following system can occur:

$$\begin{cases} \sum_{i=0}^{n} d_i b^i = pt \\ \sum_{i=0}^{n} d_i b^{n-i} = qt \end{cases}$$

Generalizations of v-palindromes in Base b

- Following calculations, the result is as follows

Theorem 9. *For a base b, where $v(p) = v(q)$ and t is co-prime to pq, such that for any $0 < \{d_2, d_3, d_4, \ldots.., d_n\} < b$, that is the range of integer digits $d_2, d_3, d_4, \ldots., d_n$ are between 0 and base b:*

The pair of equations:

$$d_0 = t\left(p - \frac{b(q - b^n p)}{b^{n-1} - b^{n+1}}\right) - \sum_{i=2}^{n}\left(b^i - \frac{1 - b^{2i}}{b^{i-1}(1 - b^2)}\right) d_i$$

$$d_1 = t\left(\frac{q - b^n p}{b^{n-1} - b^{n+1}}\right) - \sum_{i=2}^{n}\left(\frac{1 - b^{2i}}{b^{i-1}(1 - b^2)}\right) d_i$$

represents integers d_0 and d_1 based on base b and integer digits $d_2, d_3, d_4, \ldots., d_n$ such that $(d_0, d_1, d_2, \ldots..., d_n)_b$ is a v-palindrome.

Generalizations of v-palindromes in Base b

- Taking into account of already known v(5) = v(6). An example:

Lemma 19. *for a base b, where $v(5) = v(6)$ and t is co-prime to 30, for any d_2 such that $0 < d_2 < b$; the integers d_1 and d_2 of a three digit v-palindrome in base b can be found using the two equations below.*

$$d_0 = t\left(5 - \frac{b(6-5b^2)}{b-b^3}\right) - \left(b^2 - \frac{b(1-b^4)}{b-b^3}\right)d_2$$

$$d_1 = t\left(\frac{6-5b^2}{b-b^3}\right) - \left(\frac{1-b^4}{b-b^3}\right)d_2.$$

Two sequences of bases b_1 and b_2 for v-palindromes

- In Tsai original work, mentioned two sequences of v-palindromes:

$$18, 1818, 181818, 18181818, 1818181818, \ldots$$

$$18, 198, 1998, 19998, 199998, 1999998, \ldots$$

- Using the method provided in previous slide with v(18) = v(81) for an arbitrary base b_1 the following sequence is derived:

Thus, the sequence of b_1 is:

$$5832, 59492232, 5.95040132 \times 10^{11}, 5.9504131 \times 10^{15}, 5.95041322 \times 10^{19},$$
$$5.95041322 \times 10^{23}, 5.95041322 \times 10^{27}, 5.95041322 \times 10^{31}, 5.95041322 \times 10^{35}$$

Two sequences of bases b_1 and b_2 for v-palindromes

- Using the fact from previous that $b_2 = 4pq$ and previous knowledge that

$$\begin{cases} 18, 198, 19998, 19998, 199998, 1999998, ... \\ 81, 891, 8991, 89991, 899991, 8999991, ... \end{cases}$$

- Through multiplication, like so

$$h_1 = 4 \times 18 \times 81 = 5832$$

$$h_2 = 4 \times 198 \times 891 = 705672$$

$$h_3 = 4 \times 1998 \times 8991 = 71856072$$

$$h_4 = 4 \times 19998 \times 89991 = 7.19856007 \times 10^9$$

Two sequences of bases b_1 and b_2 for v-palindromes

- The following is our new sequence for b_2

$$5832, 705672, 71856072, 7.19856007 \times 10^9, 7.199856 \times 10^{11}, 7.1999856 \times 10^{13}$$
$$7.19999856 \times 10^{15}, 7.19999986 \times 10^{17}, 7.19999999 \times 10^{19}, 7.2 \times 10^{21}, 7.2 \times 10^{23},$$
$$7.2 \times 10^{25}, 7.2 \times 10^{27}, 7.2 \times 10^{29}, \ldots$$

Generalizations of v-palindromes and Connection to f-palindromes

- Define the f function:

Definition 6. *Let $f : \mathbb{N} \to \mathbb{C}$ (i.e. f is an* ***arithmetic function****). We say that $n \in \mathbb{N}_{\neq b}$ is an f**-palindrome in base** b if the following hold:*

1. *n is not a palindrome (i.e. $n \neq r_b(n)$)*
2. $$f(n) = f(r_b(n))$$

- Also, we define psi as

$$\varphi_{f,p,\delta}(\alpha) := f(p^{\alpha+\delta}) - f(p^{\alpha})$$

Generalizations of v-palindromes and Connection to f-palindromes

- Alignment

Definition 7. *Let f be an arithmetic function, $A \subseteq \mathbb{N}$, and $g : A \to \mathbb{N}$. We say that $n \in A$* ***is aligned with*** *g* ***under*** *f if $f(n) = f(g(n))$. This is said to be* ***unique*** *if $n \neq g(n)$.*

- Alignment notation

Moving forward, we want to have some notation to denote the numbers n in this fg pair. We write:

$$f \trianglerighteq g := \{n \in A | f(n) = f(g(n))\}$$

$$f \triangleright g := \{n \in f \trianglerighteq g | n \neq g(n)\}$$

Generalizations of v-palindromes and Connection to f-palindromes

- FI-criteria:

Definition 8. *Let f be an arithmetic function. We say that f satisfies the* ***Fiber Interval Criteria (FI-Criteria)*** *if the following are satisfied:*

1. *f is additive.*
2. *for all $p \in \mathbb{P}$, $\delta \geq 1$, the set $R_{f,p,\delta}$ must be finite.*
3. ***Fiber Interval Condition (FI-Condition):*** *Each fiber F of $\varphi_{f,p,\delta}$ is one of two forms, for $a, b \in \mathbb{N}_0$:*

   (a)
   $$F = \mathbb{N}_0 \cap [a, b)$$

   (b)
   $$F = \mathbb{N}_0 \setminus [a, b)$$

Generalizations of v-palindromes and Connection to f-palindromes

- FP-Criteria:

Definition 9. *Let f be an arithmetic function. We say that f satisfies the* ***Fiber Progression Criteria (FP-Criteria)*** *if f satisfies the FI-Criteria and concurrently, the FI-Condition is strengthened to the* ***Fiber Progression Condition (FP-Condition):*** *Each fiber of $\varphi_{f,p,\delta}$ is one of two forms, for $a, b \in \mathbb{N}_0$:*

1.

$$\{a, a+1, \ldots, a+b\}$$

2.

$$\{a, a+1, \ldots\}$$

Generalizations of v-palindromes and Connection to f-palindromes

- Passive numbers and passive functions:

Definition 10. *Let $n \in A$. A number $\rho \in \mathbb{N}$ is considered* ***passive to*** *n* ***in*** *g if the following hold:*

1. $n\rho \in A$
2. $\rho \mid g(n\rho)$

A function $\rho : \mathbb{N} \to \mathbb{N}$ is considered to be ***passive to*** *n* ***in*** *g if for all $k \in \mathbb{N}$, the number $\rho(k)$ is passive to n in g.*

- Divisible input:

Definition 11. *Let $\rho : \mathbb{N} \to \mathbb{N}$. This function is said to have a* ***divisible input*** *if for all prime powers p^α, there exists a number $h_{\rho,p^\alpha} \in \mathbb{N}_0$ such that for all $k \in \mathbb{N}$:*

$$p^\alpha \mid \rho(k) \iff h_{\rho,p^\alpha} \mid k$$

Generalizations of v-palindromes and Connection to f-palindromes

- Main Theorem part 1:

Theorem 10. *Let* $n \in A \subseteq \mathbb{N}$, *and suppose* $f : \mathbb{N} \to \mathbb{C}$, $g : A \to \mathbb{N}$, *and* $\rho_n : \mathbb{N} \to \mathbb{N}$ *satisfy the following conditions:*

1. f *satisfies the FI-Criteria*
2. ρ_n *is passive to* n *in* g
3. ρ_n *has divisible input and is h-non-zero*

Then there exists a number $\omega \in \mathbb{N}$ *such that for all* $k \in \mathbb{N}$*:*

$$n\rho_n(k) \in f \triangleright g \iff n\rho_n(k+\omega) \in f \triangleright g$$

Generalizations of v-palindromes and Connection to f-palindromes

- DLD-form functions:

Definition 14. *A function f is in* ***Divisible-Linear-Decomposable-Form (DLD-Form)*** *if there exists integers:*

$$0 < c_1 < c_2 < \ldots < c_q$$

$$\lambda_1, \lambda_2, \ldots, \lambda_q \neq 0$$

such that:

$$f = \sum_{j=1}^{q} \lambda_j I_{c_j}$$

Generalizations of v-palindromes and Connection to f-palindromes

- Main Theorem Part 2:

Theorem 12. *Let $n \in A \subseteq \mathbb{N}$, and suppose $f : \mathbb{N} \to \mathbb{C}$, $g : A \to \mathbb{N}$, and $\rho_n : \mathbb{N} \to \mathbb{N}$ satisfy the conditions in Theorem 10. Then the indicator function of the set:*

$$S_{n,f,g,\rho_n} = \{k \in \mathbb{N} | n\rho_n(k) \in f \triangleright g\}$$

is in DLD-Form.

- Periodically-divisible uniquely aligned

Definition 15. *If the tuple (n, f, g, ρ_n) satisfy the conclusion of the theorem, then we say that n is ρ_n-**periodically-divisible uniquely aligned with** g **under** f*

Generalizations of v-palindromes and Connection to f-palindromes

- Co-alignment:

Definition 20. *Let f and g be arithmetic functions and $h : A \to \mathbb{N}$. f and g are said to be* ***Co-Aligned with*** *h if:*

$$f \trianglerighteq h = g \trianglerighteq h$$

Generalizations of v-palindromes and Connection to f-palindromes

- To summarize:

Theorem 15. *Let $n \in A \subseteq \mathbb{N}$, and suppose $f : \mathbb{N} \to \mathbb{C}$, $g : A \to \mathbb{N}$, and $\rho_n : \mathbb{N} \to \mathbb{N}$ satisfy the following conditions:*

1. *ρ_n is passive to n in g*
2. *ρ_n has divisible input and is h-non-zero*
3. *There exists an arithmetic function f' such that f' satisfies the FI-Criteria and f is co-aligned with f'*

Then n is ρ_n-periodically-divisible uniquely aligned with g under f.

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