Towards an “Average” Version of the Birch and Swinnerton-Dyer Conjecture

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SMALL 2009 Research Program at Williams College
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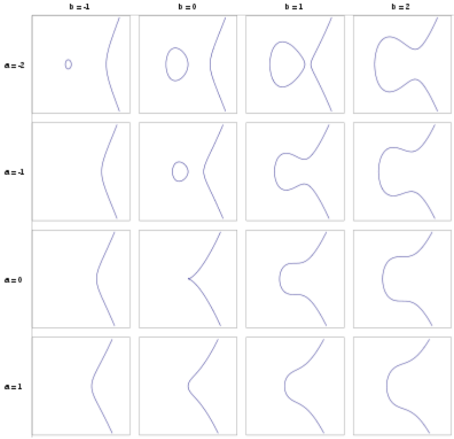
Young Mathematicians Conference
The Ohio State University, August 29, 2009

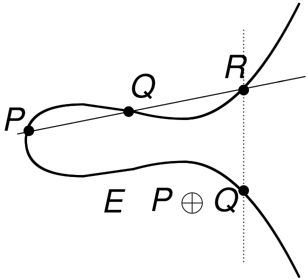
Elliptic curves: Introduction

Consider $y^2 = x^3 + ax + b$; $a, b \in \mathbb{Z}$.



Mordell-Weil Group: the Algebraic Structure of Elliptic Curves

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two rational solutions. Line $y = mx + b$ goes through and hits the curve at a third rational point on $y^2 = x^3 + ax + b$.



Addition of distinct points P and Q

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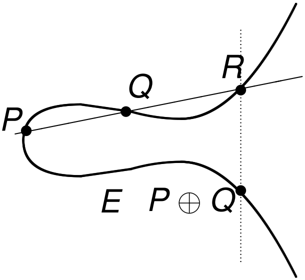

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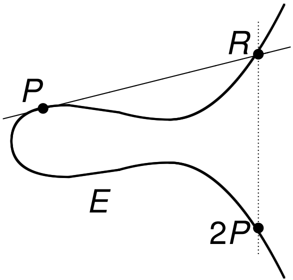

Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

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Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

General *L*-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Elliptic curve: *L*-functions

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Birch and Swinnerton-Dyer Conjecture: Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

$$L(E, s) = c\left(s - \frac{1}{2}\right)^r + \text{higher order terms}$$

Previous Results

Mestre: elliptic curve of conductor N has a zero with imaginary part at most $\frac{B}{\log\log N}$.

Expect the relevant scale to study zeros near central point to be $1/\log N_E$.

Goal: bound (from above and below) number of zeros in a neighborhood of size $1/\log N_E$ near the central point in a family.

Measures of Spacings: 1-Level Density and Families

$\phi(x)$ even Schwartz function whose Fourier Transform is compactly supported.

1-level density

$$D_f(\phi) \quad = \quad \sum_j \phi\left(L_f \gamma_{j;f}\right)$$

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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Explicit Formula for Elliptic Curves

Explicit formula: Let $\mathcal{E}$ be a family of elliptic curves with $L(s,E) = \sum_n a_E(n)/n^s$. Then

$$\frac{1}{|\mathcal{E}|}\sum_{E\in\mathcal{E}}\sum_{\gamma_E}\phi\left(\frac{\log R}{2\pi}\gamma_E\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{E}|}\sum_{E\in\mathcal{E}} P(E;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(E;\phi) = \sum_{p\nmid N_E} a_E(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

One-parameter families

$\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, $A(T)$ and $B(T)$ in $\mathbb{Z}[T]$, find solutions $(x(T), y(T)) \in \mathbb{Q}(T)^2$

For each $t \in \mathbb{Z}$ get an elliptic curve E_t:
$y^2 = x^3 + A(t)x + B(t)$.

Can construct families where group of solutions of $\mathcal{E}$ has rank r; by Silverman's specialization theorem implies each E_t has at least rank r (for t large).

Often take $T \in [R, 2R]$ with $R \to \infty$ as our family.

1-Level Density

For a family of elliptic curves $\mathcal{E}$ of rank r, we have

$$\frac{1}{|\mathcal{F}_{\mathcal{R}}|}\sum_{E\in\mathcal{F}_R}\phi\left(\tilde{\gamma}_{j,E}\frac{\log N_E}{2\pi}\right) = \left(r+\frac{1}{2}\right)\phi(0)+\widehat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

if $\widehat{\phi}(x)$ is zero for $|x| \geq \sigma_{\mathcal{E}}$.

Want $\sigma_{\mathcal{E}}$ to be large, in practice can only prove results for $\sigma_{\mathcal{E}}$ small.

Question: how many zeros ‘near’ the central point?

One-Level Density: Sketch of result

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\tilde{\gamma}_{j,f}}\phi(\tilde{\gamma}_{j,f}) = \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$N_{avg} \geq \left(r+\frac{1}{2}\right)+\frac{\widehat{\phi}(0)}{\phi(0)}+O\left(\frac{\log\log R}{\log R}\right)$$

New results for families (as conductors tend to infinity)

Theorem

Let $t_0 = C(\phi)/\sigma$, where σ is the support of $\widehat{\phi}$, $C(\phi)$ is constant depending on the choice of test function, and $N_{\rm avg}(R)$ the average number of normalized zeros in $(-t_0, t_0)$ for $T \in [R, 2R]$. Then assuming G.R.H.

$$N_{\rm avg}(R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

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Technical requirements for ϕ:

- ϕ must be even, positive in $(-t_0, t_0)$ and negative elsewhere.
- ϕ must have locally monotonically decreasing from $(0, t_0)$
- ϕ must be differentiable,
- $\widehat{\phi}$ must be compactly supported in $(-\sigma, \sigma)$.

Preliminaries

- Convolution:

$$(A * B)(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

$$\begin{aligned} \widehat{A}(y) &= \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx \\ \widehat{A''}(y) &= -(2\pi y)^2\widehat{A}(y). \end{aligned}$$

- Lemma: $\widehat{(A * B)}(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
  in particular, $\widehat{(A * A)}(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Constructing good ϕ's

- Let h be supported in $(-1, 1)$.
- Let $f(x) = h(2x/\sigma)$, so f supported in $(-\sigma/2, \sigma/2)$.
- Let $g(x) = (f * f)(x)$, so g supported in $(-\sigma, \sigma)$.
  $\widehat{g}(y) = \widehat{f}(y)^2$.
- Let $\phi(y) := (\widehat{g + \beta^2 g''})(y) = \widehat{f}(y)^2 \left(1 - (2\pi\beta y)^2\right)$.
  For β sufficiently small above is non-negative.

Constructing good ϕ's (cont)

$N_{\rm avg}(R)$ is average number of zeros in $(-t_0, t_0)$, and

$$N_{\rm avg}(R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

Want to maximize $\widehat{\phi}(0)/\phi(0)$, which is

$$\frac{(\int_0^1 h(u)^2 du) + (\frac{2\beta}{\sigma})^2 (\int_0^1 h(u) h''(u) du)}{\sigma(\int_0^1 h(u) du)^2} = R_\beta$$

Birch and Swinnerton-Dyer on "average"

By setting $R_\beta = 0$ we obtain the relationship $\beta = \mathrm{C}(h)\sigma$, allowing us an interpretation of our result in light of the Birch and Swinnerton-Dyer conjecture:

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Let $\beta = \mathrm{C}(h)\sigma$ such that $R_\beta = 0$. Then assuming G.R.H. there are on average at least $r + \frac{1}{2}$ normalized zeros within the band $\left(-\frac{1}{2\pi \mathrm{C}(h)\sigma}, \frac{1}{2\pi \mathrm{C}(h)\sigma}\right)$.

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For the function $h(x) = (1 - x^2)^2$ we obtain the result that there are at least $r + \frac{1}{2}$ normalized zeros on average within the band $\approx (-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma})$

Concrete results for certain test functions

$h(x) = 0$ for $|x| > 1$, and

- Class: $h(x) = (1 - x^{2k})^{2j}$,$(j, k \in \mathbb{Z})$
  Optimum: $h(x) = (1 - x^2)^2$
  gives interval approximately $\left(-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma}\right)$.
- Class: $h(x) = \exp\left(-1/(1 - x^{2k})\right)$,$(k \in \mathbb{Z})$
  Optimum: $h(x) = \exp\left(-1/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.558415}{\sigma}, \frac{0.558415}{\sigma}\right)$.
- Class: $h(x) = \exp\left(-k/(1 - x^2)\right)$
  Optimum: $h(x) = \exp\left(-.754212/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.552978}{\sigma}, \frac{0.552978}{\sigma}\right)$.

Bounding the number of zeros within a region from above

Theorem

For an elliptic curve with explicit formulas as above, the number of normalized zeros within $(-t_0, t_0)$ is bounded above by $(r + \frac{1}{2}) + \frac{(r+\frac{1}{2})(\psi(0)-\psi(t_0))+\hat{\psi}(0)}{\psi(t_0)}$, for all strictly positive, even test functions monotonically decreasing over $(0, \infty)$.

Thank You!