Towards an “Average” Version of the Birch and Swinnerton-Dyer Conjecture

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Elliptic curves: Introduction

Consider $y^2 = x^3 + ax + b$; $a, b \in \mathbb{Z}$. 

![Graph of elliptic curves for different values of $a$ and $b$]
Mordell-Weil Group: the Algebraic Structure of Elliptic Curves

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two rational solutions. Line $y = mx + b$ goes through and hits the curve at a third rational point on $y^2 = x^3 + ax + b$. 

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Addition of distinct points $P$ and $Q$ 

Adding a point $P$ to itself 

$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{tors} \oplus \mathbb{Z}^r$
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]
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**Functional Equation:**

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1 - s). \]
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**Riemann Hypothesis (RH):**

All non-trivial zeros have Re(s) = \( \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).
General $L$-functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \text{Re}(s) > 1.$$  

Functional Equation:

$$\Lambda(s, f) = \Lambda_\infty(s, f)L(s, f) = \Lambda(1 - s, f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma.$
Elliptic curve: $L$-functions

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{\text{p prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \mod p\}.$$
Birch and Swinnerton-Dyer Conjecture: Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

$$L(E, s) = c \left(s - \frac{1}{2}\right)^r + \text{higher order terms}$$
Previous Results

Mestre: elliptic curve of conductor $N$ has a zero with imaginary part at most $\frac{B}{\log \log N}$.

Expect the relevant scale to study zeros near central point to be $\frac{1}{\log N_E}$.

**Goal:** bound (from above and below) number of zeros in a neighborhood of size $\frac{1}{\log N_E}$ near the central point in a family.
Measures of Spacings: 1-Level Density and Families

\( \phi(x) \) even Schwartz function whose Fourier Transform is compactly supported.

1-level density

\[
D_f(\phi) = \sum_j \phi(L_f \gamma_j; f)
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2. Most of contribution is from low zeros.
3. Average over similar curves (family).
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**Katz-Sarnak Conjecture**

For a ‘nice’ family of \( L \)-functions, the \( n \)-level density depends only on a symmetry group attached to the family.
Explicit Formula for Elliptic Curves

Explicit formula: Let $\mathcal{E}$ be a family of elliptic curves with $L(s, E) = \sum_n a_E(n)/n^s$. Then

$$
\frac{1}{|\mathcal{E}|} \sum_{E \in \mathcal{E}} \sum_{\gamma_E} \phi \left( \frac{\log R}{2\pi} \gamma_E \right) = \hat{\phi}(0) + \frac{1}{2} \phi(0) - \frac{1}{|\mathcal{E}|} \sum_{E \in \mathcal{E}} P(E; \phi) + O \left( \frac{\log \log R}{\log R} \right)
$$

$$
P(E; \phi) = \sum_{p \nmid N_E} a_E(p) \hat{\phi} \left( \frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}.
$$
One-parameter families

\[ E : y^2 = x^3 + A(T)x + B(T), \quad A(T) \text{ and } B(T) \text{ in } \mathbb{Z}[T], \text{ find solutions } (x(T), y(T)) \in \mathbb{Q}(T)^2 \]

For each \( t \in \mathbb{Z} \) get an elliptic curve \( E_t \):
\[ y^2 = x^3 + A(t)x + B(t). \]

Can construct families where group of solutions of \( E \) has rank \( r \); by Silverman’s specialization theorem implies each \( E_t \) has at least rank \( r \) (for \( t \) large).

Often take \( T \in [R, 2R] \) with \( R \to \infty \) as our family.
1-Level Density

For a family of elliptic curves $\mathcal{E}$ of rank $r$, we have

$$\frac{1}{|\mathcal{F}_R|} \sum_{E \in \mathcal{F}_R} \phi \left( \tilde{\gamma}_{j,E} \frac{\log N_E}{2\pi} \right) = \left( r + \frac{1}{2} \right) \phi(0) + \hat{\phi}(0) + O \left( \frac{\log \log R}{\log R} \right)$$

if $\hat{\phi}(x)$ is zero for $|x| \geq \sigma_{\epsilon}$.

Want $\sigma_\epsilon$ to be large, in practice can only prove results for $\sigma_\epsilon$ small.

**Question:** how many zeros ‘near’ the central point?
One-Level Density: Sketch of result

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{\tilde{\gamma}_j, t} \phi(\tilde{\gamma}_j, t) = \left( r + \frac{1}{2} \right) \phi(0) + \hat{\phi}(0) + O \left( \frac{\log \log R}{\log R} \right)
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One-Level Density: Sketch of result

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\frac{1}{|F|} \sum_{f \in F} \sum_{\tilde{\gamma}_j, f} \phi(\tilde{\gamma}_j, f) = \left( r + \frac{1}{2} \right) \phi(0) + \hat{\phi}(0) + O \left( \frac{\log \log R}{\log R} \right)
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\[
\frac{1}{|F|} \sum_{f \in F} \sum_{|\tilde{\gamma}_j, f| \leq t_0} \phi(\tilde{\gamma}_j, f) \geq \left( r + \frac{1}{2} \right) \phi(0) + \hat{\phi}(0) + O \left( \frac{\log \log R}{\log R} \right)
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N_{avg} \geq \left( r + \frac{1}{2} \right) + \frac{\hat{\phi}(0)}{\phi(0)} + O \left( \frac{\log \log R}{\log R} \right)
\]
New results for families (as conductors tend to infinity)

**Theorem**

Let $t_0 = C(\phi)/\sigma$, where $\sigma$ is the support of $\hat{\phi}$, $C(\phi)$ is constant depending on the choice of test function, and $N_{\text{avg}}(R)$ the average number of normalized zeros in $(-t_0, t_0)$ for $T \in [R, 2R]$. Then assuming G.R.H.

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Technical requirements for $\phi$:

- $\phi$ must be even, positive in $(-t_0, t_0)$ and negative elsewhere.
- $\phi$ must have locally monotonically decreasing from $(0, t_0)$
- $\phi$ must be differentiable,
- $\hat{\phi}$ must be compactly supported in $(-\sigma, \sigma)$. 
Preliminaries

- **Convolution:**
  \[(A \ast B)(x) = \int_{-\infty}^{\infty} A(t)B(x - t)dt.\]

- **Fourier Transform:**
  \[\hat{A}(y) = \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy} dx\]
  \[\hat{A}''(y) = -(2\pi y)^2 \hat{A}(y).\]

- **Lemma:** \((A \ast B)(y) = \hat{A}(y) \cdot \hat{B}(y);\)
  in particular, \((A \ast A)(y) = \hat{A}(y)^2 \geq 0\) if \(A\) is even.
Constructing good \( \phi \)'s

- Let \( h \) be supported in \((-1, 1)\).

- Let \( f(x) = h(2x/\sigma) \), so \( f \) supported in \((-\sigma/2, \sigma/2)\).

- Let \( g(x) = (f \ast f)(x) \), so \( g \) supported in \((-\sigma, \sigma)\).
  \[
  \hat{g}(y) = \hat{f}(y)^2.
  \]

- Let \( \phi(y) := (g + \beta^2 g'')(y) = \hat{f}(y)^2 \left(1 - (2\pi \beta y)^2\right)\).
  For \( \beta \) sufficiently small above is non-negative.
Constructing good $\phi$’s (cont)

$N_{\text{avg}}(R)$ is average number of zeros in $(-t_0, t_0)$, and

$$N_{\text{avg}}(R) \geq \left( r + \frac{1}{2} \right) + \frac{\hat{\phi}(0)}{\phi(0)} + O \left( \frac{\log \log R}{\log R} \right).$$

Want to maximize $\frac{\hat{\phi}(0)}{\phi(0)}$, which is

$$\frac{\left( \int_{0}^{1} h(u)^2 du \right) + \left( \frac{2\beta}{\sigma} \right)^2 \left( \int_{0}^{1} h(u)h''(u) du \right)}{\sigma \left( \int_{0}^{1} h(u) du \right)^2} = R_\beta$$
By setting $R_{\beta} = 0$ we obtain the relationship $\beta = C(h)\sigma$, allowing us an interpretation of our result in light of the Birch and Swinnerton-Dyer conjecture:
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**Theorem**

Let $\beta = C(h)\sigma$ such that $R_\beta = 0$. Then assuming G.R.H. there are on average at least $r + \frac{1}{2}$ normalized zeros within the band $\left(-\frac{1}{2\pi C(h)\sigma}, \frac{1}{2\pi C(h)\sigma}\right)$. 

For the function $h(x) = (1 - x^2)^2$ we obtain the result that there are at least $r + \frac{1}{2}$ normalized zeros on average within the band $\approx \left(-0.551329\sigma, 0.551329\sigma\right)$. 
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Concrete results for certain test functions

\[ h(x) = 0 \text{ for } |x| > 1, \text{ and} \]

- **Class:** \( h(x) = (1 - x^{2k})^{2j}, (j, k \in \mathbb{Z}) \)
  - Optimum: \( h(x) = (1 - x^2)^2 \)
  - Gives interval approximately \( \left( -\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma} \right) \).

- **Class:** \( h(x) = \exp \left( -\frac{1}{1 - x^{2k}} \right), (k \in \mathbb{Z}) \)
  - Optimum: \( h(x) = \exp \left( -\frac{1}{1 - x^2} \right) \)
  - Gives approximately \( \left( -\frac{0.558415}{\sigma}, \frac{0.558415}{\sigma} \right) \).

- **Class:** \( h(x) = \exp \left( -\frac{k}{1 - x^2} \right) \)
  - Optimum: \( h(x) = \exp \left( -\frac{0.754212}{1 - x^2} \right) \)
  - Gives approximately \( \left( -\frac{0.552978}{\sigma}, \frac{0.552978}{\sigma} \right) \).
Bounding the number of zeros within a region from above

**Theorem**

For an elliptic curve with explicit formulas as above, the number of normalized zeros within \((-t_0, t_0)\) is bounded above by \((r + \frac{1}{2}) + \frac{(r+\frac{1}{2})(\psi(0) - \psi(t_0)) + \hat{\psi}(0)}{\psi(t_0)}\), for all strictly positive, even test functions monotonically decreasing over \((0, \infty)\).
Thank You!