Generalized harmonic estimates for the n-level density of L-functions

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SMALL REU 2022, Williams College

Young Mathematicians Conference
The Ohio State University, 13 August 2022

Properties of Sequences

Important Example: Let $\mathcal{P} = (2, 3, 5, 7, \ldots)$ be the sequence of primes.

- How many are there $\leq x$?
- Prime gaps $p_{n+1} - p_n$?
- Which is more "likely" : $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$?

The Big Picture

The Riemann Zeta function $\zeta(s)$ has formula for $\Re(s) > 1$:

$$\zeta(s) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

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Riemann noticed that

$$\Lambda(s) \ = \ \Gamma_{\mathbb{R}}(s)\zeta(s) \ =: \ \Lambda(1-s)$$

is a functional equation equation that extends $\Lambda(s)$ to the entire complex plane and preserves differentiability.

The Big Picture

Using residue calculus, Riemann relates

$$\textbf{Primes}\ \mathcal{P} \quad \longleftrightarrow \quad \textbf{Zeroes}\ \text{of}\ \zeta(s).$$

We have related two sequences:

$$\mathcal{P} \;=\; (2,3,5,7,\dots) \quad \longleftrightarrow \quad (\pm\gamma^{(1)},\pm\gamma^{(2)},\pm\gamma^{(3)},\dots)$$

where $\zeta(\frac{1}{2} \pm i\gamma^{(j)}) \;=\; 0$.

Riemann Hypothesis: $\gamma^{(j)} \in \mathbb{R}$ for all j.

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Our original questions about primes:

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Can be answered by sequences of zeroes:

- Riemann (1859): $\pi(x) \ = \ \mathrm{Li}(x) + O(\sqrt{x}\log x)$*.
- Zhang (2013): $\liminf_{n\to\infty} \left(p_{n+1} - p_n\right) \ < \ 70,000,000.$
- Rubinstein–Sarnak (1994): More likely* $p \equiv 3 \bmod 4$.

*Results about zeroes are conjectural.

Our Picture

Using residue calculus, Iwaniec–Luo–Sarnak (2000) relate

Fourier Coefficients $a_f(p)$ $\longleftrightarrow$ **Zeroes** of $L(f,s)$

where $f : \mathbb{H} \to \mathbb{C}$ is a "modular form."

We have related two sequences:

$$\mathcal{F} = (a_f(2), a_f(3), a_f(5), \ldots) \quad \longleftrightarrow \quad (\pm\gamma^{(1)}, \pm\gamma^{(2)}, \pm\gamma^{(3)}, \ldots)$$

where $L(f, \frac{1}{2} \pm i\gamma^{(j)}) = 0$.

Generalized Riemann Hypothesis: $\gamma^{(j)} \in \mathbb{R}$ for all j.

Our Object of Interest

We are interested in zeroes arising from **Modular Forms**.

Definition

In this talk, the modular forms we consider are *holomorphic cuspidal newforms* of weight k and level $N = 1$.

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

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The first non-trivial example is when $k = 12$. $\left(q = e^{2\pi i z}\right)$

$$\begin{aligned}\Delta(z) &:= q\prod_{n=1}^{\infty}(1-q^n)^{24} \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots\end{aligned}$$

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We use the bridge between arithmetic and analysis **in the other direction:**

Riemann: Analysis of Zeroes $\Longrightarrow$ Distribution of Primes.

ILS: Analysis of Zeroes $\Longleftarrow$ Distribution of Coefficients.

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There is no known algorithm for the geometric rank.

Studying Low-lying zeros

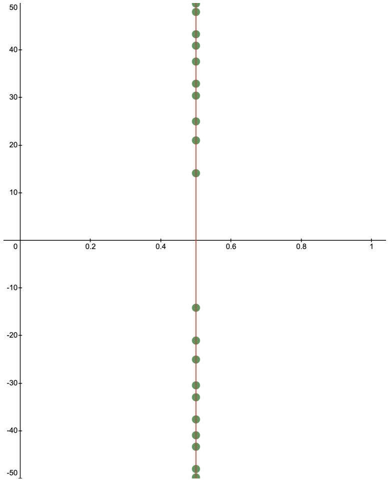

Figure: Zeros of the ζ function along the critical strip.

We want to study the distribution of low-lying zeros near the central point.

Test Functions

Procedure: We weight contributions of zeros using test functions $\phi : \mathbb{R}^n \to [0, \infty)$ which are even, Schwartz, and $\widehat{\phi}$ having compact support.

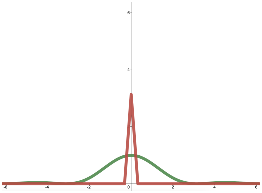

Figure: $\widehat{\phi}$ has smaller support $\Longrightarrow$ *less precise* information about low lying zeros

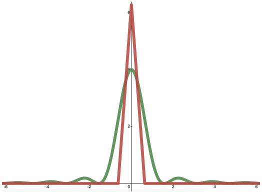

Figure: $\widehat{\phi}$ has larger support $\Longrightarrow$ *more precise* information about low lying zeros

Density

Definition (1-level density)

The **1-level density** of $L(f,s)$ is

$$D(f;\phi) := \sum_{j=-\infty}^{\infty} \phi\left(\frac{\log c_f}{2\pi}\gamma_f^{(j)}\right),$$

where $\frac{1}{2}+i\gamma_f^{(j)}$ are zeros of $L(f,s)$ and c_f is the conductor of $L(f,s)$.

Result from ILS

Theorem (ILS)

For $\text{supp}(\widehat{\phi}) \subset (-2, 2)$, we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k\leqslant K} \frac{4\pi^2}{k-1} \sum_{f\in \mathrm{H}_k^\star(1)} \omega_f \mathrm{D}(f;\phi) \ = \ \int_{-\infty}^{\infty} \phi(x) W_O(x) dx$$

where

$$\omega_f \ := \ \mathrm{L}^{-1}\left(\text{sym}^2(f), 1\right)$$

are harmonic weights and

$$W_O(x) \ := \ 1 + \frac{1}{2}\delta_0(x)$$

is a smooth weighting function arising from Random Matrix Theory, where $\mathcal{G} = O$ denotes the orthogonal type.

Hypothesis S

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For any $x \geqslant 1$, $c \geqslant 1$, and a with $(a, c) = 1$,

$$\sum_{p \leqslant x, p \equiv a(c)} e(2\sqrt{p}/c) \ll_{\varepsilon} x^{\frac{1}{2}+\varepsilon}.$$

where $e(z) = e^{2\pi i z}$.

Theorem (ILS)

Assuming Hypothesis S, $\mathrm{supp}(\widehat{\phi}) \subset (-22/9, 22/9)$, we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k \leqslant K} \frac{4\pi^2}{k-1} \sum_{f \in \mathrm{H}_k^\star(1)} \omega_f \mathrm{D}(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W_O(x) dx$$

2-level Density

The 2-level density is

$$\begin{aligned} D(f;\phi_1,\phi_2) &:= \sum_{\substack{j_1,j_2 \\ j_1 \neq \pm j_2}} \phi_1\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_1)}\right)\phi_2\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_2)}\right) \\ &= D(f;\phi_1)D(f;\phi_2) - 2D(f;\phi_1\phi_2) \end{aligned}$$

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$$= D(f;\phi_1)D(f;\phi_2) - 2D(f;\phi_1\phi_2)$$

RMT predicts that

RMT

$$\frac{1}{|H_k^*(N)|}\sum_{f\in H_k^*(N)} D(f;\phi_1,\phi_2) \approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(x)\phi_2(y)W_{O,2}(x,y)\,dx dy$$

$$-2\int_{-\infty}^{\infty}\phi_1(t)\phi_2(t)W_O(t)\,dt.$$

2-level density

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Repeating the steps in ILS, one can show that the above is true when

$$\mathrm{supp}(\widehat{\phi_1})\times\mathrm{supp}(\widehat{\phi_2}) \subset (-\sigma_1,\sigma_1)\times(-\sigma_2,\sigma_2)$$

and $\sigma_1+\sigma_2 \leq 2$.

A New Hypothesis

Consider the following analogue of *Hypothesis S* for the 2-level density.

Hypothesis T

$$\sum_{\substack{p_1 \leq x_1 \\ p_1 \equiv a_1 (c)}} \sum_{\substack{p_2 \leq x_2 \\ p_2 \equiv a_2 (c)}} e\left(\frac{2\sqrt{p_1 p_2}}{c}\right) \ll_\varepsilon c^A (x_1 x_2)^{\alpha+\varepsilon}$$

Conjecture (MM–)

Assuming Hypothesis T, we can extend the support of $(\widehat{\phi_1}, \widehat{\phi_2})$ in the 2-level density asymptotic formula.

References

Thank you!

This research was conducted as part of the **2022 SMALL REU program** at Williams College, and was funded by **NSF Grant DMS1947438**.

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