# L-function Lecture 1: PANTHers 2021

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# Welcome to L-fns for PANTHERS

Riemann Zeta Fn

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad Re(s) > 1$$

$$= \prod_{p \text{ primes}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Fund Thm of Arith: $n = p_1^{r_1} \cdots p_k^{r_k}$

Geo Series Formula:

$|r| < 1 \qquad 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$

$$\lim_{s \to 1^+} \zeta(s) = \sum 1/n \text{ diverges}$$

$$\log(1-x) \simeq -x$$

$\Rightarrow \exists\ \infty$ many primes

$$\sum_{n \le x} \frac{1}{n^s} \simeq \prod_{p \le x} \left(1 - 1/p^s\right)^{-1}$$

PAULOU: Take logs, set $s=1$

$$\log \underbrace{\sum_{n \le x} \frac{1}{n}} \simeq -1 \sum_{p \le x} \log\left(1 - \frac{1}{p}\right) \simeq \sum_{p \le x} \frac{1}{p}$$

$$\log \log x \simeq \sum_{p \le x} 1/p$$

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2) = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1}$$

$\pi^2/6$ transcendental

↳ irrational

if only finite # of primes this rational

$\zeta(2) \notin \mathbb{Q} \Rightarrow \#\{p \le x\} \gg \log\log\log x$

why BAD?

(1) PNT: $\pi(x) \simeq x/\log x$

(2) Euclid: $p_1 \cdots p_n + 1 \Rightarrow \pi(x) \gg \log\log x$

(3) My proof uses result on $LCM(1, \ldots, n)$ which relies on the PNT

# Generating Functions

Saw Binet from Fibonacci

$$\sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} = \prod_p L_{p,f}(1/p^s)$$

$$\frac{d}{ds} \log L(f,s) = \frac{d}{ds} \sum_p \log L_{p,f}(1/p^s)$$

PAULOVIAN RESPONSES

↳ Product: Think logs

↳ Complex, always good to do a contour integral of $\int \frac{f'(z)}{f(z)} dz$

$$\frac{f'(z)}{f(z)} = \frac{d}{dz} \log f(z)$$

objects WANT to study

object CAN study

Explicit Formulas

Eigenvalues: $A\vec{v} = \lambda \vec{v}$ $\lambda$ evalue, $\vec{v} \neq \vec{0}$ evector

Random Matrix Theory: $(a_{ij})$

$$\mathrm{Tr}(A^k) = \sum \lambda_i(A^k)$$

Explicit Formula (Evalue Trace Lemma)

Hope: If know enough about something it's uniquely determined.

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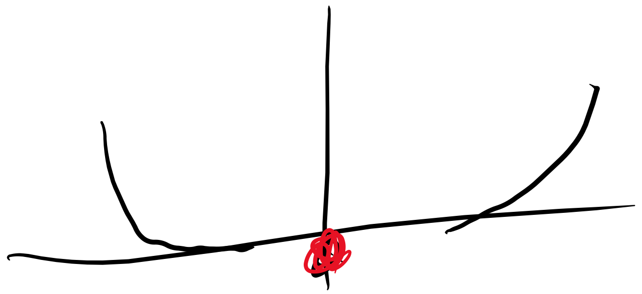
$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

L'Hopital: $f^{(n)}(0) = 0$



$$T_{\infty} f)(x) \equiv 0$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

Prob dist: non-neg, $\int$ to 1

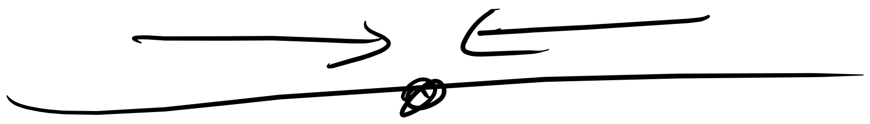
$k^{th}$ moment $\displaystyle\int_{-\infty}^{\infty} x^k p(x)\,dx$

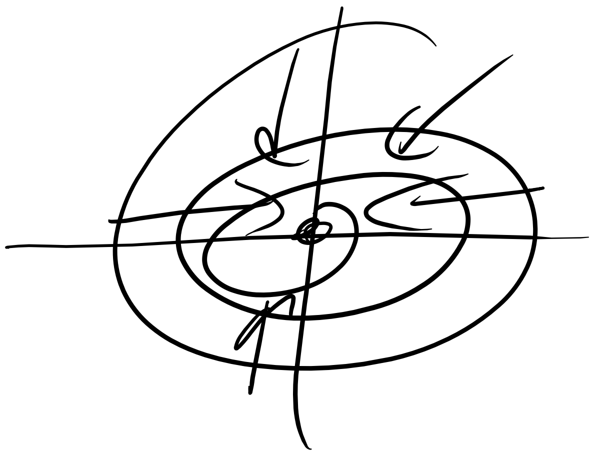
Hope: moments uniquely determine $p$

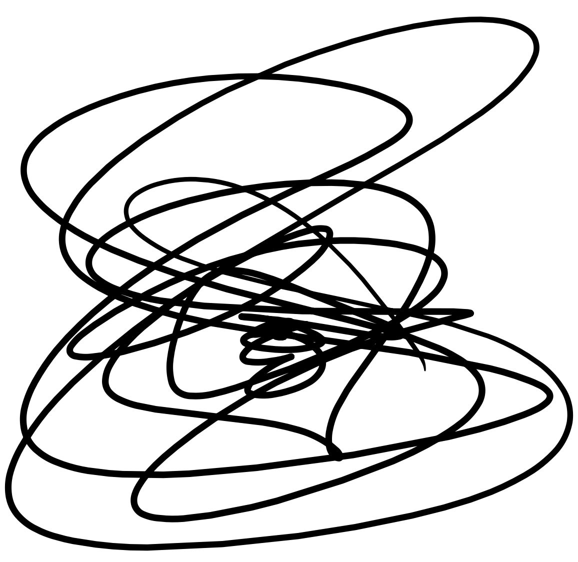
(k integer)

$\frac{1}{n}\sum \lambda_i(A^k)$ $k^{th}$ moment of evalues

# Differentiable







Next:
Complex
Analysis