L-function Lecture 2: PANTHers 2021
COMPLEX ANALYSIS IN AN HOUR

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https://web.williams.edu/Mathematics/sjmiller/public_html/

An Invitation to Modern Number Theory

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Complex Analysis: Math 372

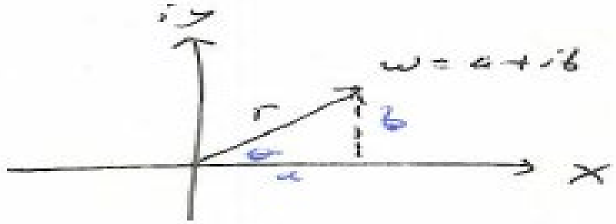
https://web.williams.edu/Mathematics/sjmiller/public_html/372Fa17/index.htm

Videos of lectures: https://web.williams.edu/Mathematics/sjmiller/public_html/videoclasses/index.htm https://web.williams.edu/Mathematics/sjmiller/public_html/372Fa17/coursenotes/Math302_LecNotes_Intro.pdf

QUICK INTRODUCTION TO COMPLEX ANALYSIS

Numbers:

- $\mathbb{N}$ natural #s ~~integers~~: $\{0, 1, 2, \dots\}$ or $\{1, 2, \dots\}$
- $\mathbb{Z}$ integers: $\{\dots, -1, 0, 1, \dots\}$ (zahl = number in German)
- $\mathbb{Q}$ rationals: $\{p/q : p, q \in \mathbb{Z}, q \neq 0\}$ (quotient)
- $\mathbb{R}$ reals
- $\mathbb{C}$ complex: $\{x + iy : x, y \in \mathbb{R}\}$, $i^2 = -1$



General Properties

$z = x + iy$, $\quad w = a + ib$, $\quad i^2 = -1$

↳ can write $z = re^{i\theta}$ with $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$

$\bar{z} = x - iy$ (complex conjugate)

$z + w = (x + a) + i(y + b)$, $\quad zw = (xa - yb) + i(xb + ya)$

$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

↳ Extra credit: show cannot order the complex numbers (i.e, no generalization of $<$ from $\mathbb{R}$)

↳ $|z+w| \leq |z| + |w|$ (Triangle inequality)

↳ Commutativity: $z+w = w+z$

Associativity: $(z+w)+q = z+(w+q)$

Distributivity: $z(w+q) = zw + zq$

↳ See quaternions and octonions for generalization

What's the Big Deal: Real vs Complex Analysis

- Real Analysis
  - 1 variable: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

    or $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \leftrightarrow \lim_{x \to x_0} \frac{|f(x) - f(x_0) - f'(x_0)(x - x_0)|}{|x - x_0|} = 0$

  - 2 variable: f is differentiable at $\vec{x}_0$ if

    $$\lim_{\vec{x} \to \vec{x}_0} \frac{|f(\vec{x}) - f(\vec{x}_0) - (\nabla f)(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)|}{\|\vec{x} - \vec{x}_0\|} = 0$$

    - Note it is possible for $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ to exist without having f differentiable; if, however, partials $\partial f/\partial x$ and $\partial f/\partial y$ are continuous then f is differentiable

      - Extra credit: find a function f st partials exist but f is not differentiable

    - Note limit must exist along any path, not just along coordinate axes:  or 

↳ Note limit must exist along <u>any</u> path, not just along co-ordinate axes!  or 

↳ If you know Green's Thm / Stokes' Thm: review!

• COMPLEX DIFFERENTIABLE: $f: \mathbb{C} \to \mathbb{C}$ is complex diff if $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists for all paths.

Real vs Complex (continued)

Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and let also $g: \mathbb{C} \to \mathbb{C}$ be a complex differentiable function, and let $h: \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function.

Property	(Real diff) True for f	(Complex diff) True for g
Function is infinitely differentiable	NO	YES
function equals its Taylor Series in some neighborhood of point where differentiable	NO	YES
the function may be bounded without being identically constant	YES	NO
The line integral of our function along a closed curve must be zero	NO*	YES

New behavior: shame cannot visualize $g: \mathbb{C} \to \mathbb{C}$ as "4-dimensional"
↳ 2^{nd} point important in probability. 4^{th} point is Green/Stokes' Thm

* For h not f

Review: Convergence

Seq $\{z_n\}_{n=0}^{\infty}$ converges to a limit L if $\lim_{n\to\infty} |z_n - L| = 0$

Should be able to prove the following:

(1) L is unique

(2) $\{z_n\}_{n=0}^{\infty}$ converges to L if and only if (iff) $\{Re(z_n)\}_{n=0}^{\infty}$ converges to $Re(L)$ and $\{Im(z_n)\}_{n=0}^{\infty}$ converges to $Im(L)$.

Hint: The triangle inequality is useful: $|\alpha \pm \beta| \leq |\alpha| + |\beta|$

Key definition for the course:

<u>COMPLEX DIFFERENTIABILITY</u>

A complex valued function f on an open $\Omega \subset \mathbb{C}$ is holomorphic (ie, complex differentiable) at $z_0 \in \Omega$ if $\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exists, in which case we denote the limit $f'(z_0)$.

VERY STRONG CONDITION AS CAN DO **ANY** PATH

EXAMPLES

$$f(z) = z: \quad f'(z_0) = \lim_{h \to 0} \frac{z_0 + h - z_0}{h} = \lim_{h \to 0} 1 = 1$$

$$f(z) = z^2: \quad f'(z_0) = \lim_{h \to 0} \frac{(z_0+h)^2 - z_0^2}{h}$$

$$= \lim_{h \to 0} \frac{z_0^2 + 2z_0 h + h^2 - z_0^2}{h}$$

$$= \lim_{h \to 0} (2z_0 + h) = 2z_0$$

More generally, any finite poly $\sum_{n=0}^{N} a_n z^n$

$$f(z) = \bar{z}: \quad f'(z_0) = \lim_{h \to 0} \frac{\overline{z_0 + h} - \bar{z}_0}{h}$$

$$= \lim_{h \to 0} \bar{h} / h$$ Does not exist!

↳ take h in $\mathbb{R}$ and get 1

take h in $i\mathbb{R}$ and get -1

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Article Talk

Cauchy–Riemann equations

From Wikipedia, the free encyclopedia

Complex differentiability [edit]

Suppose that

$$f(z) = u(z) + i \cdot v(z)$$

is a function of a complex number $z = x + iy$. Then the complex derivative of f at a point z_0 is defined by

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

provided this limit exists.

If this limit exists, then it may be computed by taking the limit as $h \to 0$ along the real axis or imaginary axis; in either case it should give the same result. Approaching along the real axis, one finds

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z_0).$$

On the other hand, approaching along the imaginary axis,

$$\lim_{\substack{\eta \to 0 \\ \eta \in \mathbb{R}}} \frac{f(z_0 + i\eta) - f(z_0)}{i\eta} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

The equality of the derivative of *f* taken along the two axis is

$$i\frac{\partial f}{\partial x}(z_0) = \frac{\partial f}{\partial y}(z_0),$$

which are the Cauchy–Riemann equations (2) at the point z_0.

The Cauchy–Riemann equations on a pair of real-valued functions of two real variables *u*(*x*,*y*) and *v*(*x*,*y*) are the two equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \textbf{(1a)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \textbf{(1b)}$$

Cauchy-Riemann Eqs

Write $f: \Omega \to \mathbb{C}$ as $f(z) = u(x,y) + i v(x,y)$, $z = x+iy$

f holo $\Rightarrow$ $f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ for any path

↳ take path h real: $h = (h_1, h_2)$ or $h_1 + i h_2$ so on this path $h_2 = 0$. Get

$$f'(z_0) = \lim_{\substack{h_1 \to 0 \\ h_1 \text{ real}}} \frac{f(z_0+h_1) - f(z_0)}{h_1}$$

write $f(x,y)$ for $f(z)$: abuse of notation

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0+h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}(z_0)$$

↳ now take path $h = i h_2$ and find

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0+h_2) - f(x_0, y_0)}{i h_2} = \frac{1}{i} \frac{\partial f}{\partial y}(z_0)$$

Thus $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}$

As $f(x,y) = u(x,y) + i v(x,y)$, find

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equate real and imaginary parts

$$\boxed{\text{Cauchy-Riemann Eqs} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

CAUCHY-RIEMANN EQ (CONT)

Define $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

PROP 2.3 : f holo $\Longrightarrow$ $\partial f/\partial \bar{z} = 0$ and $f'(z_0) = \dfrac{\partial f}{\partial z} = 2\dfrac{\partial u}{\partial z}$

IMPORTANT CONVERSE (SEE PAGE 13 for Proof)

Thm 2.4 : $f = u + iv$ complex valued fn defined on open $\Omega \subset \mathbb{C}$. If u, v cont diff and satisfy the Cauchy-Riemann Eqs on Ω then f is holo on Ω and $f'(z) = \partial f/\partial z$.

Connection with Stokes' Thm

Say $\vec{F}(x,y) = (P(x,y), Q(x,y))$ and have

Then $$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial\Omega} \vec{F}\cdot d\vec{s} = \int_{\partial\Omega} P\,dx + Q\,dy$$

Say f is holomorphic, so $f = u + iv$

By Cauchy-Riemann Eqs: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$dz = dx + i\,dy$ $\quad f = u + iv$ so

$f\,dz = (u+iv)(dx+i\,dy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy)$

Apply Stokes' (actually Green's) Thm to each

Connection with Stokes' Thm

$$\int_{\partial\Omega} f\,dz = \int_{\partial\Omega} (u+iv)(dx+i\,dy)$$

$$= \int_{\partial\Omega} u\,dx - v\,dy + i\int_{\partial\Omega} v\,dx + u\,dy$$

$$= \iint_{\Omega} \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

↳ $= 0$ by Cauchy-Riemann Eqs

Give Additional, Optional lecture on proof of Stokes' Thm

Thus "expect" $\int_{\partial\Omega} f\,dz = 0$ for holomorphic f!

↳ first instance that Complex differentiability is very special

(2.3) Power Series

Thm 2.6: Power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holo fn in the disc of convergence. Deriv of f is also a power series and is obtained by differentiating term by term:

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^n$$ and has the same radius of conv

↳ Corollary: applying above again and again, find a power series is infinitely diff in its disc of convergence.

https://web.williams.edu/Mathematics/sjmiller/public_html/372Fa17/index.htm

Summary

↳ Cauchy's Thm: γ simple closed curve separating plane into inside and outside, and f holomorphic in open Ω containing γ, then $\int_\gamma f(z)\,dz = 0$

↳ sometimes write $\oint_\gamma f(z)\,dz$ to emphasize closed curve

Applications

↳ Many problems in math/phys can be reduced to evaluating an integral.

Examples

- Number Theory: Goldbach: $\int_0^1 \left(\sum_{p \leq N} e^{2\pi i p x}\right)^5 e^{-2\pi i n x}\,dx$ gives the # ways of writing n as a sum of 5 primes at most N
- Probability: Normalization constants: $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$
- Fresnel integrals: $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$
- Cauchy → if holomorphic then infinitely differentiable

  ↳ leads to Liouville: f entire and bounded → f constant

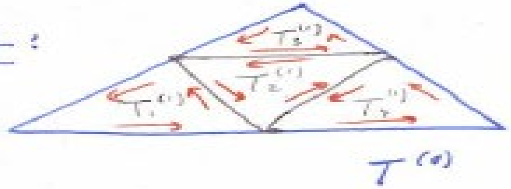
  ↳ leads to a proof of the Fund Thm of Algebra

Thm: Goursat

OPEN $\Omega \subset \mathbb{C}$, T TRIANGLE $\subset \Omega$, f HOLOMORPHIC IN Ω, THEN $\int_T f(z)\,dz = 0$.

<u>Comment</u>: instead of triangles could use rectangles; convenient as triangularization

<u>Proof</u>:



All interior integrals cancel

$$\int_{T^{(0)}} f\,dz = \sum_{j=1}^{4} \int_{T_j^{(1)}} f\,dz$$

(don't use i as dummy variable!)

Thus for at least one j, $\left|\int_{T^{(0)}} f\,dz\right| \le 4 \left|\int_{T_j^{(1)}} f\,dz\right|$

↳ JUSTIFY!

Call this triangle $T^{(1)}$, and continue

Get chain $T^{(0)} \supset T^{(1)} \supset T^{(2)} \ldots$

Triangles similar, $\operatorname{diam} T^{(n)} = 2^{-n} \operatorname{diam} T^{(0)}$

$\operatorname{perim} T^{(n)} = 2^{-n} \operatorname{perim} T^{(0)}$

$\operatorname{area} T^{(n)} = 4^{-n} \operatorname{area} T^{(0)}$

Thus there exists a unique z_0 in all $T^{(n)}$ ($\exists!\, z_0 \in T^{(n)}$ for all n)

$$\left|\int_{T^{(0)}} f\,dz\right| \le 4^n \left|\int_{T^{(n)}} f\,dz\right|$$

↳ Why useful (does this? On $T^{(n)}$ f is approx constant

Proof of Goursat (cont)

Now use f is holomorphic!

↳ f is differentiable at z_0, so for n large in $T^{(n)}$ have

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)$$

with $\lim_{z \to z_0} \psi(z) = 0$

Now $\int_{T^{(n)}} f\,dz = \int_{T^{(n)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z-z_0)\,dz$

$+ \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$

Note first two are integrals of polynomials on closed curve and thus vanish as have primitives z and $\frac{(z-z_0)^2}{2}$

Last integral is $\leq \max_{z \in T^{(n)}} |\psi(z)| \cdot \text{perim } T^{(n)}$

$= \max_{z \in T^{(n)}} |\psi(z)| \cdot 4^{-n} \text{perim } T^{(0)}$

Thus $\left| \int_{T^{(0)}} f\,dz \right| \leq 4^n \cdot \max_{z \in T^{(n)}} |\psi(z)| \cdot 4^{-n} \text{perim } T^{(0)}$

or $\left| \int_{T^{(0)}} f\,dz \right| \leq \text{perim } T^{(0)} \cdot \max_{z \in T^{(n)}} |\psi(z)|$

as $n \to \infty$, $\max_{z \in T^{(n)}} |\psi(z)| \to 0$ and thus $\int_{T^{(0)}} f\,dz = 0$ ▨

Comments: Note use of topology

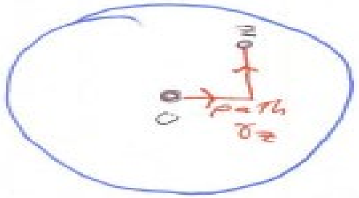
Note approx complicated fn f with linear function + error

SECTION 2: LOCAL EXISTENCE OF PRIMITIVES AND CAUCHY IN A DISK

THM: A holomorphic fn in an open disc has a primitive in the disc

Proof: Recall F is a primitive for f if F is holo on Ω and $F' = f$

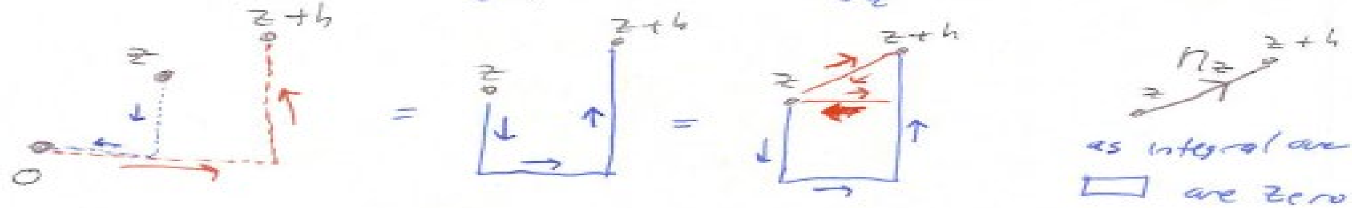
↳ Wlog assume disc centred at origin.



Define $F(z) := \int_{\gamma_z} f(w)\,dw$

Claim: F holo in Ω (our disc) and $F' = f$

↳ $F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,dw - \int_{\gamma_z} f(w)\,dw$



as integral are $\triangle$ and $\square$ are zero as f holo

Thus $F(z+h) - F(z) = \int_{\eta_z} f(w)\,dw$

↳ By continuity: $f(w) = f(z) + \psi(w)$, $\psi(w) \to 0$ as $w \to z$

$$F(z+h) - F(z) = \int_{\eta_z} f(z)\,dw + \int_{\eta_z} \psi(w)\,dw$$

$$= f(z)\, w \Big|_z^{z+h} + \int_{\eta_z} \psi(w)\,dw$$

$$= f(z)\, h + \int_{\eta_z} \psi(w)\,dw$$

$$\Rightarrow \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \le \left| \int_{\eta_z} \psi(w)\,dw \right| / |h|$$

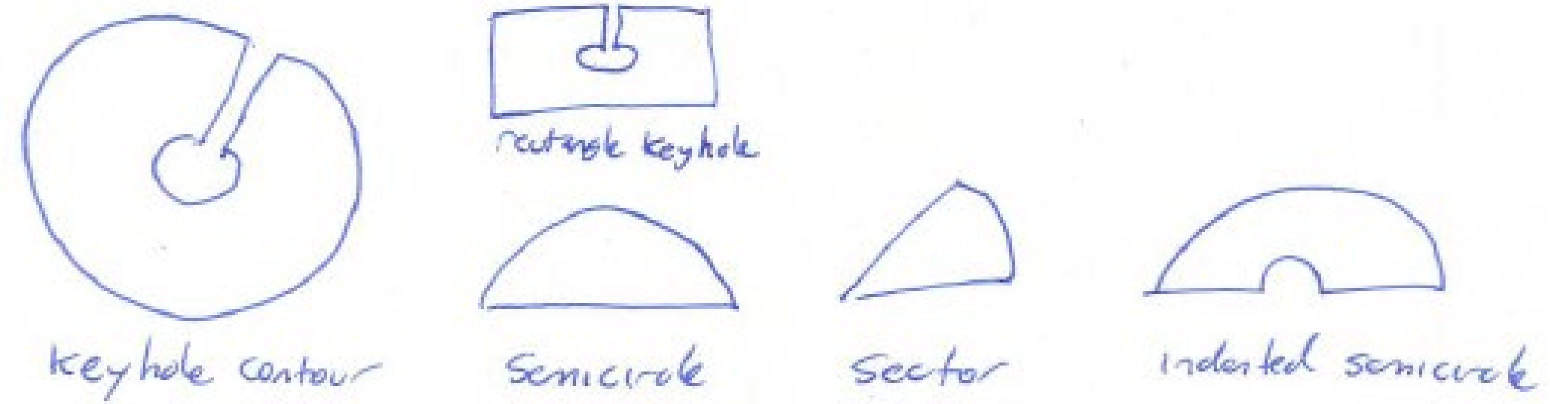
$$\le \frac{\max_{w \in \eta_z} |\psi(w)| \cdot |h|}{|h|} \to 0$$

CAUCHY'S THM: f holomorphic in a disc and γ closed curve in disc, then $\oint_\gamma f(z)\,dz = 0$

Proof: Previous theorem shows f has a primitive.
Done by Cor 3.3 of Chapter 1.

DEFN: TOY CONTOUR: ANY CLOSED CURVE WHERE NOTION OF INTERIOR/EXTERIOR IS CLEAR.

Exs:

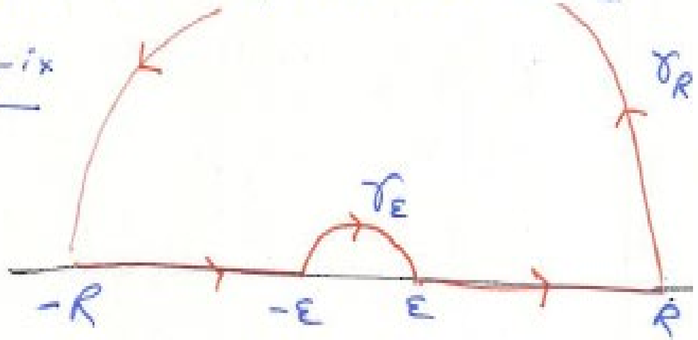


Example 2: $\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\, dx = \pi$

First note reasonable: decays like $1/x^2$, near origin looks like $\frac{1}{2}$

Write $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

Use following contour



$\boxed{\text{EQ } \phi}$ $\int_{-R}^{-\varepsilon} + \int_{\gamma_\varepsilon} + \int_{\varepsilon}^{R} + \int_{\gamma_R} f(z)\,dz = 0$ if f holo on contour

Idea: $\int_{\gamma_\varepsilon} \to$ known as $\varepsilon \to 0$, $\int_{\gamma_R} \to 0$ as $R \to 0$

would leave us with $\int_{\varepsilon \le |x| \le R}$ of what we desire

PROBLEM: $e^{iz} = e^{ix-y}$ is small as $y \to \infty$ but e^{-iz} blows up

If use $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ need to use upper half plane contour for e^{ix} and lower half plane for e^{-ix}

$\boxed{\text{Alternative: Take } f(z) = \frac{1-e^{iz}}{z^2} \text{ and take real part of } (\phi)}$

On γ_ε have $\int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,dz = \int_{\gamma_\varepsilon} \frac{1-(1+iz+E(z))}{z^2}\,dz$ with $|E(z)| \le C|z|^2$

$$= \int_{\gamma_\varepsilon} -i\frac{dz}{z} + \underbrace{\int_{\gamma_\varepsilon} \frac{E(z)}{z^2}\,dz}_{\substack{\text{integrand} \le C \\ \text{bounded by } C\cdot\pi\varepsilon \to 0}}$$

$$-i\int_{\pi}^{0} \frac{\varepsilon i e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}} = -i\cdot i\cdot(-\pi)$$

On γ_R have $\left|\frac{1-e^{iz}}{z^2}\right| \le \frac{1+|e^{ix}|e^{-y}}{|z|^2} \le \frac{1+e^{-y}}{|z|^2} \le \frac{2}{R^2}$ as $|z| = R$ on γ_R. As $\text{length}(\gamma_R) = \pi R$, we have

$$\left|\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,dz\right| \le \frac{2}{R^2}\cdot \pi R \to 0$$

Thus $\int_{\varepsilon \le |x| \le R} \frac{1-e^{ix}}{x^2}\,dx - \pi +$ Errors tending to zero as $\varepsilon \to 0$ and $R \to \infty$ $= 0$

so $\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\,dx = \pi$; result follows by taking real parts.

THM: f holo in open Ω containing disc D with boundary C (with positive orientation). Then for any $z \in D$:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Will have many applications

↳ note integral need not vanish as $\frac{f(\zeta)}{\zeta - z}$ is not holomorphic in ζ

↳ Lang: If Greeks were intelligent would have invented $\frac{1}{2\pi i}$ as constant

↳ Note integral need not vanish as $\frac{f(\zeta)}{\zeta - z}$ is not holomorphic in ζ

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Proof: Here δ = width of corridor

ε = radius of small disc

As $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holo in interior of $\Gamma_{\delta,\varepsilon}$,

$$\int_{\Gamma_{\delta,\varepsilon}} F(\zeta)\, d\zeta = 0$$

As $\delta \to 0$ integral are two parts of corridor cancel by continuity

$$\Rightarrow 0 = \int_C F(\zeta)\, d\zeta - \int_{C_\varepsilon} F(\zeta)\, d\zeta = 0, \quad C_\varepsilon \text{ positive orientation}$$

so $$\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Exploit the fact that f is holomorphic and ζ and z are close on C_ε:

$$\frac{f(\zeta)}{\zeta - z} = \underbrace{\frac{f(\zeta) - f(z)}{\zeta - z}}_{\text{bounded as } f \text{ holo}} + \frac{f(z)}{\zeta - z}$$

Sec 4: Cauchy's Integral Formula (cont)

As $\dfrac{f(\zeta)-f(z)}{\zeta-z}$ is bounded on C_ε, the integral of this piece is bounded by $B\cdot 2\pi\varepsilon$ for some constant B.

Note $\displaystyle\int_{C_\varepsilon}\frac{f(z)}{\zeta-z}\,d\zeta$: Let $\zeta = z+\varepsilon e^{i\theta}$, $d\zeta = i\varepsilon e^{i\theta}\,d\theta$

$$= f(z)\int_0^{2\pi}\frac{i\varepsilon e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}} = 2\pi i f(z) \quad \underline{\text{indep of } \varepsilon!}$$

Thus as $\varepsilon\to 0$ obtain $\displaystyle\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta-z}\,d\zeta = \cancel{\frac{1}{2\pi i}}\, f(z)$ ▨

Corr: f holo on open Ω then f is infinitely complex differentiable, and for any $z\in\Omega$, if C is a circle contained in Ω containing z,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta$$

Definition 3.1.12 (Zero, Pole, Order, Residue). *Assume f has a convergent Taylor series expansion (see §A.2.3) about z_0:*

$$f(z) \;=\; a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots \;=\; \sum_{m=n}^{\infty} a_m(z-z_0)^m, \quad (3.9)$$

with $a_n \neq 0$. Thus n is the location of the first non-zero coefficient. If $n > 0$ we say f has a zero of order n at z_0; if $n < 0$ we say f has a pole of order $-n$ at z_0. If $n = -1$ we say f has a simple pole with residue a_{-1}. We denote the order of f at z_0 by $\operatorname{ord}_f(z_0) = n$.

Definition 3.1.13 (Meromorphic Function). *We say f is meromorphic at z_0 if the Taylor expansion about z_0 converges for all z close to z_0. In particular, there is a disk about z_0 of radius r and an integer n_0 such that for all z with $|z-z_0| < r$,*

$$f(z) \;=\; \sum_{n \geq n_0} a_n(z-z_0)^n. \quad (3.10)$$

If f is meromorphic at each point in a disk, we say f is meromorphic in the disk. If $n_0 \geq 0$, we say f is ***analytic****.*

Theorem 3.2.8. *Let $r > 0$ and $n \in \mathbb{Z}$. Then*

$$\frac{1}{2\pi i}\int_{\theta=0}^{2\pi}(re^{i\theta})^n ire^{i\theta}d\theta \;=\; \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.40}$$

Sketch of the proof. In (3.40) we have $e^{i(n+1)\theta} = \cos((n+1)\theta) + i\sin((n+1)\theta)$ (de Moivre's Theorem). If $n = -1$ the integrand is 1 and the result follows; if $n \neq -1$ the integral is zero because we have an integral number of periods of sin and cos (see the following exercise). □

Exercise 3.2.9. *For $m > 0$, show that $\sin(mx)$ and $\cos(mx)$ are periodic functions of period $\frac{2\pi}{m}$ (i.e., $f(x + \frac{2\pi}{m}) = f(x)$). If m is a positive integer, these functions have an integral number of periods in $[0, 2\pi]$.*

If we consider all $z \in \mathbb{C}$ with $|z| = r > 0$, this set is a circle of radius r about the origin. As only θ varies, we have $dz = izd\theta$, and the integral in (3.40) becomes

$$\frac{1}{2\pi i}\oint_{|z|=r} z^n dz \;=\; \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.41}$$

The symbol $\oint$ denotes that we are integrating about a closed curve, and $|z| = r$ states which curve. Note the answer is independent of the radius.

Exercise 3.2.10. *Prove*

$$\frac{1}{2\pi i}\oint_{|z-z_0|=r}(z - z_0)^n dz \;=\; \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.42}$$

Exercise 3.2.10. *Prove*

$$\frac{1}{2\pi i}\oint_{|z-z_0|=r}(z-z_0)^n dz \;=\; \begin{cases} 1 & \text{if } n=-1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.42}$$

We state, but do not prove, the following theorems on **contour integration** (see, for example, [Al, La6, SS2]:

Theorem 3.2.11. *Let $f(z)$ be a meromorphic function (see Definition 3.1.13) in a disk of radius r about z_0, say*

$$f(z) \;=\; \sum_{n\geq n_0} a_n(z-z_0)^n. \tag{3.43}$$

If $f(z)$ is not identically zero, for r sufficiently small f has no poles in this disk except possibly at z_0; it has a pole if $n_0<0$. Then

$$\frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)dz \;=\; a_{-1}. \tag{3.44}$$

Sketch of the proof. If we could interchange summation and integration, we would have

$$\frac{1}{2\pi i}\oint f(z)dz \;=\; \frac{1}{2\pi i}\oint\sum_{n\geq n_0} a_n(z-z_0)^n dz \;=\; \sum_{n\geq n_0} a_n\,\frac{1}{2\pi i}\oint(z-z_0)^n dz. \tag{3.45}$$

The only non-zero integral is when $n=-1$, which gives 1. The difficulty is in justifying the interchange. $\square$

Theorem 3.2.12. *Let $f(z)$ be a meromorphic function in a disk of radius r about the origin, with finitely many poles (at $z_1, z_2, \ldots, z_N$). Then*

$$\frac{1}{2\pi i}\oint_{|z|=r} f(z)dz \;=\; \sum_{n=1}^{N} a_{-1}(z_n), \tag{3.46}$$

where $a_{-1}(z_n)$ is the ***residue*** *(the -1 coefficient) of the Taylor series expansion of $f(z)$ at z_n.*

The proof is similar to the standard proof of Green's Theorem (see Theorem A.2.9). In fact, the result holds not just for integrating over circular regions, but over any "nice" region. *All that matters are whether there are any poles of f in the region, and if so, what their residues are.*

Exercise 3.2.13 (Logarithmic Derivative). *Let $f(z) = a_n(z - z_0)^n$, $n \neq 0$. Show*

$$\frac{f'(z)}{f(z)} \;=\; \frac{na_n}{z - z_0}. \tag{3.47}$$

Note $\frac{f'(z)}{f(z)} = \frac{d\log f(z)}{dz}$. In particular, this implies

$$\frac{1}{2\pi i}\oint_{|z-b|=r} \frac{f'(z)}{f(z)}dz \;=\; \begin{cases} na_n & \text{if } z_0 \text{ is in the disk of radius } r \text{ about } b \\ 0 & \text{otherwise.} \end{cases} \tag{3.48}$$

Theorem 3.2.14. *Let $f(z)$ be a meromorphic function on a disk of radius r about z_0. Assume the only zeros or poles of $f(z)$ inside a disk of radius r about b are $z_1, \ldots, z_N$. Then*

$$\frac{1}{2\pi i} \oint_{|z-b|=r} \frac{f'(z)}{f(z)} dz = \sum_{n=1}^{N} \operatorname{ord}_f(z_n). \tag{3.52}$$

(See Definition 3.1.12 for the definition of $\operatorname{ord}_f(z_n)$*.)*

Exercise[(h)] 3.2.15 (Important). *For $\Re s > 1$ show that*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_n \frac{\Lambda(n)}{n^s}. \tag{3.53}$$

A common method to deal with products is to take the logarithm (see Theorem 10.2.4 for another example) because this converts a product to a sum, and we have many methods to understand sums.

The idea is as follows: we integrate each side of (3.53) over the perimeter of a box (the location is chosen to ensure convergence of all integrals). For convenience, we first multiply each side by $\frac{x^s}{s}$, where x is an arbitrary parameter that we send to infinity. This will yield estimates for $\pi(x)$. We then have

$$\oint_{\text{perimeter}} \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds = -\oint_{\text{perimeter}} \sum_n \frac{\Lambda(n)}{n^s} \frac{x^s}{s} ds. \tag{3.54}$$

Using results from complex analysis, one shows that $\oint \frac{(x/n)^s}{s} ds$ is basically 1 if $n < x$ and 0 otherwise. For the left hand side, we get contributions from the zeros and poles of $\zeta(s)$ in the region. $\zeta(s)$ has a pole at $s = 1$ of residue 1. Let ρ range over the zeros of $\zeta(s)$. One needs to do some work to calculate the residues of $\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}$ at the zeros and poles of $\zeta(s)$; the answer is basically $\frac{x^\rho}{\rho}$. Combining the pieces and multiplying by -1 yields

$$x - \sum_{\rho:\zeta(\rho)=0} \frac{x^\rho}{\rho} = \sum_{n \le x} \Lambda(n). \tag{3.55}$$

The $x = x^1$ is from the pole of $\zeta(s)$ at $s = 1$. The above is known as an Explicit Formula (to be discussed in detail in §3.2.5). Note $|x^\rho| = x^{\Re \rho}$. If we knew all the zeros had $\Re \rho = \frac{1}{2}$, then the left hand side of (3.55) is approximately $x + O(x^{1/2})$ (we are ignoring numerous convergence issues in an attempt to describe the general features). One can pass from knowledge of $\sum_{n<x} \Lambda(x)$ to $\pi(x) = \sum_{n<x} 1$: first

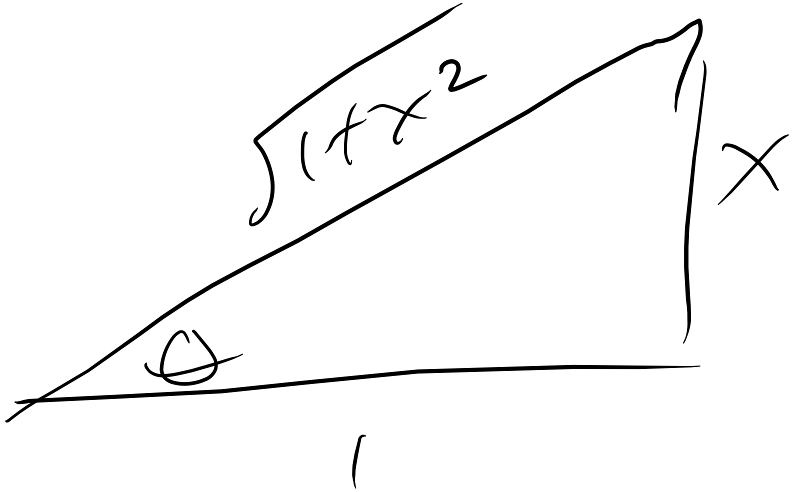
Thm: f, g holomorphic and $f(z_n) = g(z_n)$ for a seq $z_n \to z^*$. Then $f = g$

Special case: $g(z) = 0$, gives $f(z) = 0$

FALSE FOR REAL FNS

$$f(x) = \begin{cases} x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$ Cauchy Distribution



$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$$\tan(\arctan(x)) = x$$

$$\tan'(\arctan x)\cdot \arctan'(x) = 1$$

$$\arctan'(x) = \frac{1}{\tan'(\arctan x)}$$

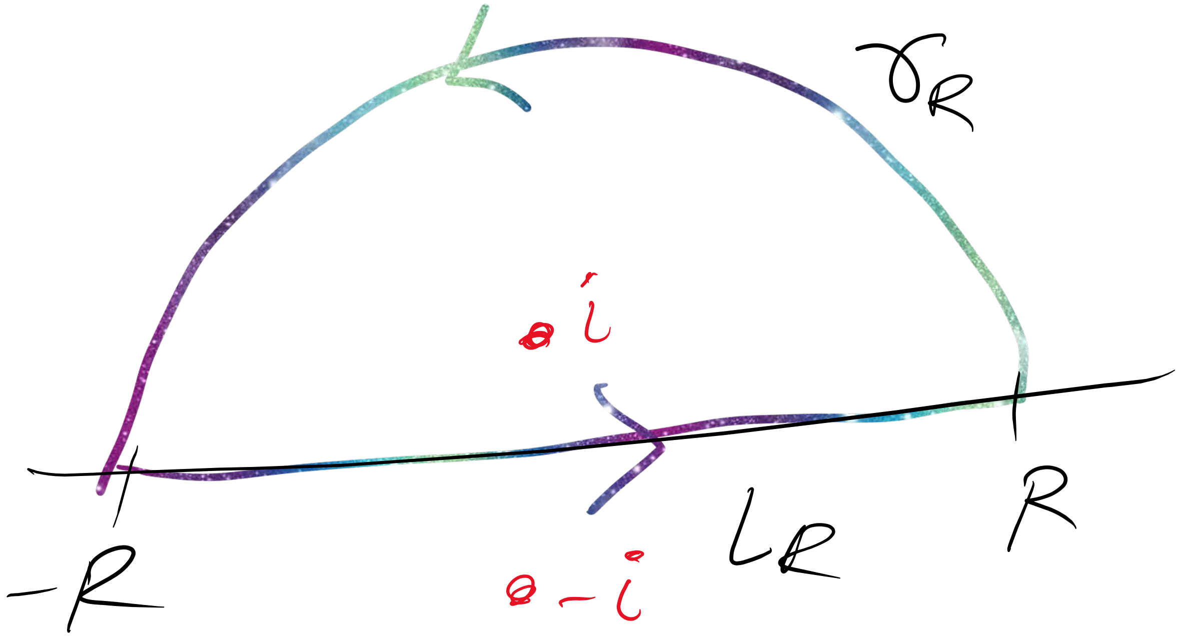
$$= \cos^2(\arctan x) = \frac{1}{1+x^2}$$

$\theta = \arctan x$

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

What about $\dfrac{1}{1+x^4}$?

Consider $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



$\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = a_{-1}(f,i) = \frac{1}{2i}$

$\oint_\gamma f(z)\,dz = \pi$

$f(z) = \frac{1}{1+z^2} = \frac{1}{z-i}\,\frac{1}{z+i}$

$a_{-1}(f,i) = \frac{1}{2i}$ $\quad \frac{1}{z+i} = \text{const} + \cancel{\ast}(z-i)$

On γ_R: $|f(z)| \le \frac{1}{R^2}$

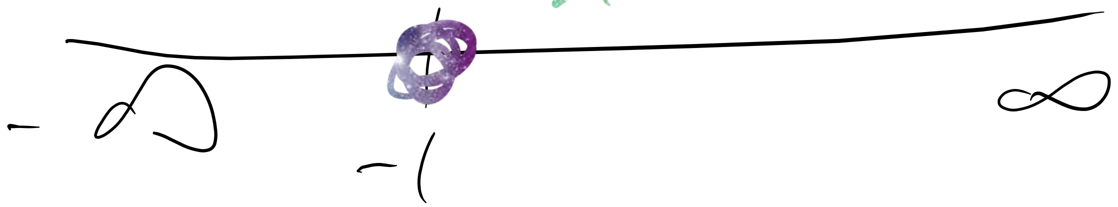
Length$(\gamma_R) = \pi R$

Contribution of γ_R is at most $\pi R / R^2 \to 0$

$\Longrightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$

Exercise

$$\int_{-\infty}^{\infty} \frac{1}{1+x^n}\,dx$$



Blows up if n odd!

n easy is even

n hard if odd

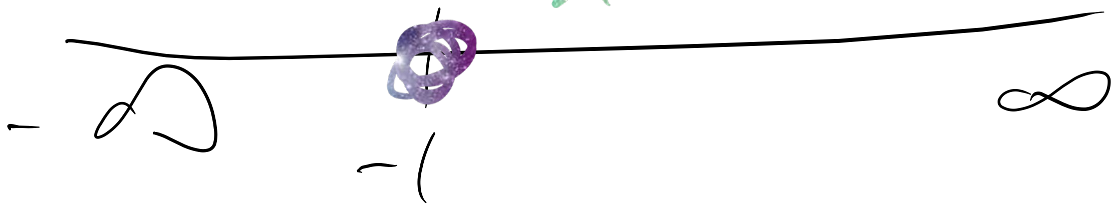
$\frac{1}{1+x^n}$ is not odd if n odd

Danger point!

$x = -1$

Exercise

$$\int_{\cancel{-\infty}}^{\infty} \frac{1}{1+x^n}\,dx$$



Blows up if n odd!

Fix $-\infty$ to 0

n easy if even

n hard if odd

$\frac{1}{1+x^n}$ is not odd if n odd

$$\int_0^\infty \frac{1}{1+x^n}\,dx$$

Danger point!

$x = -1$