L-function Lecture 3: PANTHers 2021 Comments on L-Function Talks

Miller: sjm1@williams.edu

https://web.williams.edu/Mathematics/sjmiller/public_html/

Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Michigan2012Part1.pdf

Part II (Convolving families, cusp forms: slides here): http://youtu.be/vJz6W24tDik (slides part 2)
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Michigan2012Part2.pdf

More talks, as well as papers, available here:
https://web.williams.edu/Mathematics/sjmiller/public_html/

Eigenvalue Trace Lemma

$$Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A)$$

Proof: Trivial if A is diagonal

Trivial if A is upper or lower Triangular

↳ Show any matrix can be triangularized

Given A $\exists S^{-1} = S^{-1}(A)$ st $S^{-1}AS = T$

$$\lambda(S^{-1}AS) = \lambda(A)$$

$$Tr(S^{-1}AS) = Tr(ASS^{-1}) = Tr(A)$$

ξ vs ζ

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s \quad (\mathrm{Re}(s) > 1) \quad = \prod_p (1 - 1/p^s)^{-1}$$

$$\binom{5}{2} = 10 = \frac{5!}{2!\,3!} = \frac{5\cdot 4}{2\cdot 1}$$

$$\binom{2}{5} = 0 = \frac{2!}{5!\,(-3)!} = \frac{2}{120\cdot(-3)!}$$

so $(-3)! =$ "infinity"

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(n+1) = n!$$

$n \geq 0$ int

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\,dx = \int_0^{\infty} e^{-x} x^s \frac{dx}{x}$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

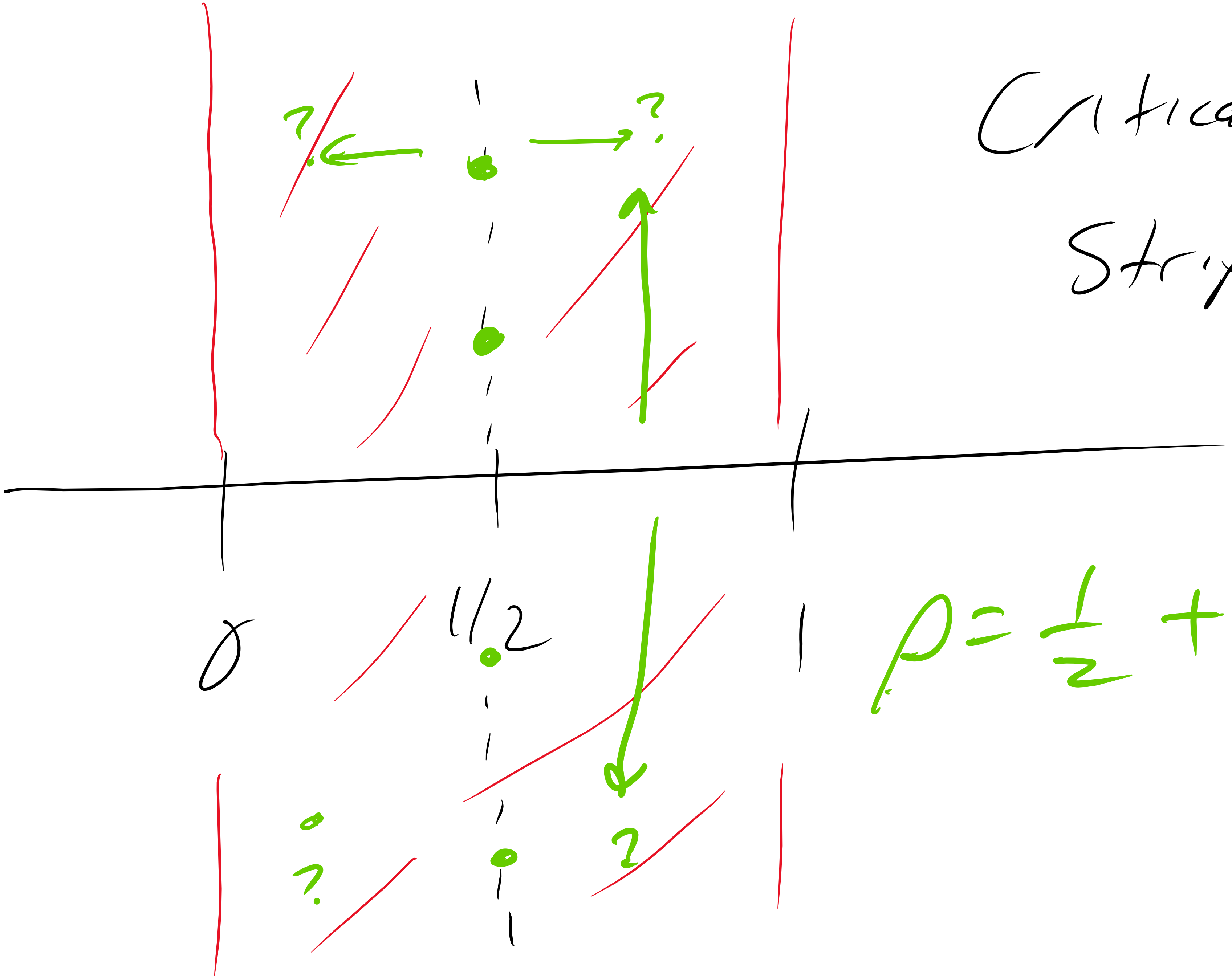
$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

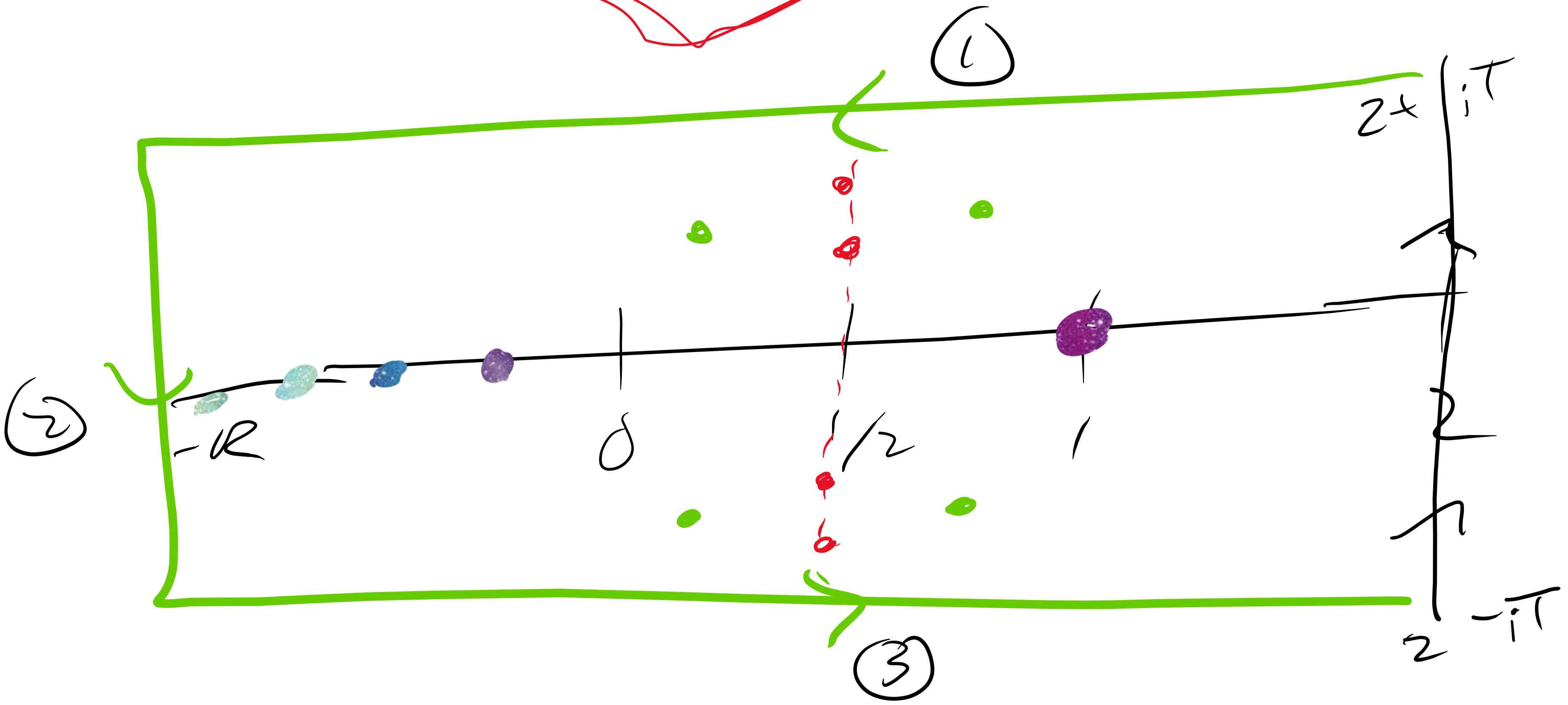
Critical Strip



$$\rho = \frac{1}{2} + i\gamma, \; \gamma \in \mathbb{R}$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \overbrace{\int \left(\frac{x}{p}\right)^s \frac{ds}{s}}^{\substack{1 \text{ if } p<x \\ 0 \text{ if } p>x}}.$$



limit as $T, R \to \infty$

hope is $\int_{1,2,3} \to 0$

Contour Integration:

pole: $x^1/1$

$\sum_{p \le x} \log p \approx x + \text{Error}$

$$\underbrace{\int -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}\, ds}_{} \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

Contribution from a zero $\rho = \frac{1}{2} + i\gamma$

$\zeta(s) = a_n(s-\rho)^n + \ldots$

$\zeta'(s) = n a_n (s-\rho)^{n-1} + \ldots$

$\frac{\zeta'(s)}{\zeta(s)} \approx \frac{n}{s-\rho}$ if zero or pole of order n

$\frac{\zeta'(s)}{\zeta(s)} = \frac{n}{s-\rho} + \text{holom.}$

zeros contribute $-\frac{x^\rho}{\rho}$: if RH true : Abs value is $\frac{x^{1/2}}{|\rho|}$

Q: if we know $\sum_{p \le x} \log p \approx x$ what is $\sum_{p \le x} 1$?

$2^k = x$

$k = \log_2 x$

$\pi(x) = \#\{\text{primes } p \le x\}$: $\pi(x) \le x$

$\mathcal{P}(x) = \#\{p^k \le x, \ p \text{ prime}, \ k \text{ integer}\}$

Prime number Thm: $\pi(x) \approx x/\log x$, better $\int_2^x \frac{dt}{\log t}$

$\mathcal{P}(x) = \pi(x) + \pi(x^{1/2}) + \pi(x^{1/3}) + \dots + \pi(x^{1/\log_2 x})$

So $|\mathcal{P}(x) - \pi(x)| \le \pi(x^{1/2}) + \pi(x^{1/3}) \log_2 x$

$\le x^{1/2} + x^{1/3} \cdot \log_2 x$

$\ll x^{1/2}$

Study $\sum_{p \le x} 1$ Given $\sum_{p \le x} \log p \approx x$

Could Study $\sum_{n \le x} \Lambda(n) \approx x$

here $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$

Show $\sum_{p \le x} \log p \approx \sum_{n \le x} \Lambda(n)$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

Goal: pass from $\sum_{p \le x} \log p$ or $\sum_{n \le x} \Lambda(n)$ to $\sum_{p \le x} 1$

$x \sim$

Show this is about $x/\log x$

HW: try to do this

$$p \le \frac{x}{\log^2 x} \qquad \frac{x}{\log^2 x} \le p \le x$$

\# primes here is at most $\frac{x}{\log^2 x}$

Contributes $x/\log x$ to $\sum_{p \le x} \log p$

How big is $\log p$ here?

$\log x - \log\log^2 x \le \log p \le \log x$

$\log p \simeq \log x$

Contributes $\simeq [\log x]\left[\pi(x) - \pi\left(\frac{x}{\log^2 x}\right)\right]$

Gives $\log x \cdot \pi(x) \simeq x$

What about negative even integer zeros from Zeta?

$$\sum_{n=1}^{\infty} -x^{2n}/2n = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(x^2)^n}{n}$$

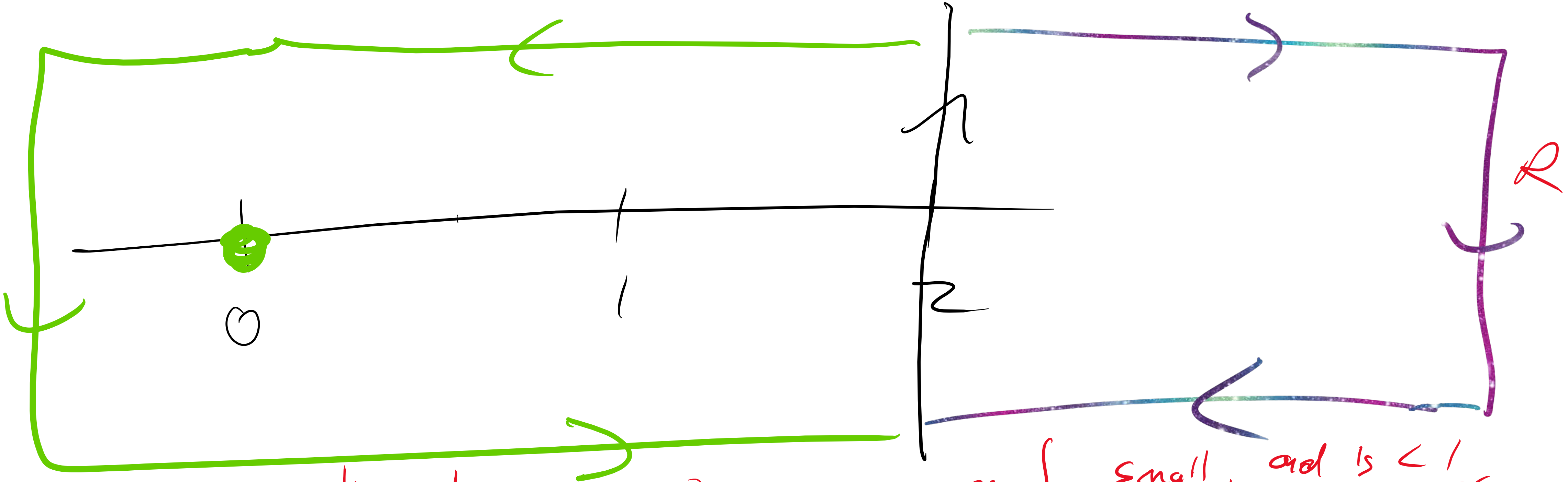
$$= \frac{1}{2}\log(1-x^2)$$

AS $\quad \log(1-u) = -\sum_{n=1}^{\infty}\frac{u^n}{}$

Study $\displaystyle\int_{Re(s)=2} \left(\frac{x}{p}\right)^s \frac{ds}{s}$

Complete the Contour

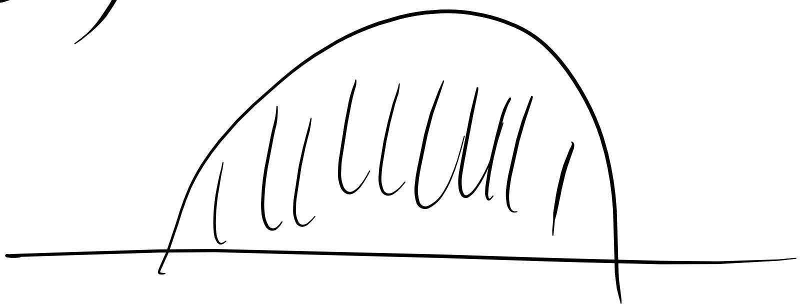
Depending on is $|x/p| > 1$ or < 1 get different answer



if $|x/p| < 1$ shift right as $\int_R$ small, and is < 1 further $p > x$

RMT: $\frac{1}{N} \sum \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$

Density of States



Gaps / Spacings b/w eigenvalues

$$\frac{1}{N-1} \delta\left(x - \frac{\lambda_{i+1}(A) - \lambda_i(A)}{g(N)}\right)$$