L-function Lecture 4: PANTHers 2021
Comments on L-Function Talks: II

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https://web.williams.edu/Mathematics/sjmiller/public_html/

Part I (Classical RMT, Intro L-fns, Dirichlet): http://youtu.be/2PuUbk6gUMM (slides: part 1)
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Michigan2012Part1.pdf

Part II (Convolving families, cusp forms: slides here): http://youtu.be/vJz6W24tDik (slides part 2)
https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/Michigan2012Part2.pdf

More talks, as well as papers, available here:
https://web.williams.edu/Mathematics/sjmiller/public_html/

Seq $\alpha_1, \alpha_2, \alpha_3, \ldots$ increasing order

Normalize to have ave gap of size 1

Ex: primes: 2, 3, 5, 7, 11, 13, ...

Gaps: 1, 2, 2, 4, 2, ...

$\pi(x) = \#\{\text{primes } p \leq x\} \approx \frac{x}{\log x}$

So if $p \sim x$, next prime is at approx $x + \log x$

Study NOT p but $p/\log p$

$\zeta(s)$: if $s = \frac{1}{2} + iT$ next zero is about $\frac{1}{\log T}$ away, so

look at mult by $\log T$

How often $\alpha_i - \alpha_k \in I$ 2-level correlation

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

$\alpha_1, \alpha_2, \alpha_3, \ldots$

$\alpha_{j_1} - \alpha_{j_2} \in I$ with $j_1, j_2 \leq N$

not nece adjacent

inclusion/exclusion

n-level corr $\Rightarrow$ spacing distribution

$\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$

$B = I_1 \times I_2$

$\alpha_{j_2} - \alpha_{j_1} \in I_1$

$\alpha_{j_3} - \alpha_{j_2} \in I_2$

Imagine $I_1 = I_2$

Symmetric about zero

$j_3 = \cancel{j_2}\, j_1$

If happens, get a

2-level lurking

can remove with incl/excl

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

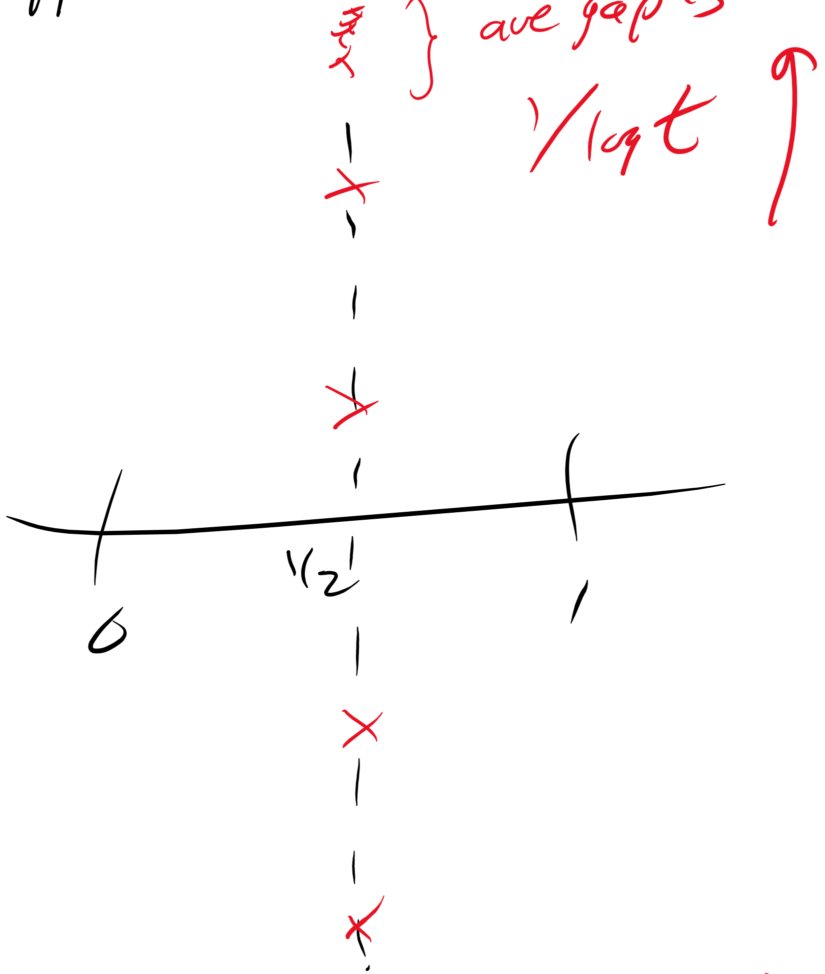
$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

2-level: sum over all zeros, subtract $j_1 = \pm j_2$

$j_1 = j_2$ $\quad \sum_j \phi_1\left(L_f \gamma_f^{(j_1)}\right) \phi_2\left(L_f \gamma_f^{(j_2)}\right) = \sum_j (\phi_1 \phi_2)\left(L_f \gamma_f^{(j_1)}\right)$

1-level density

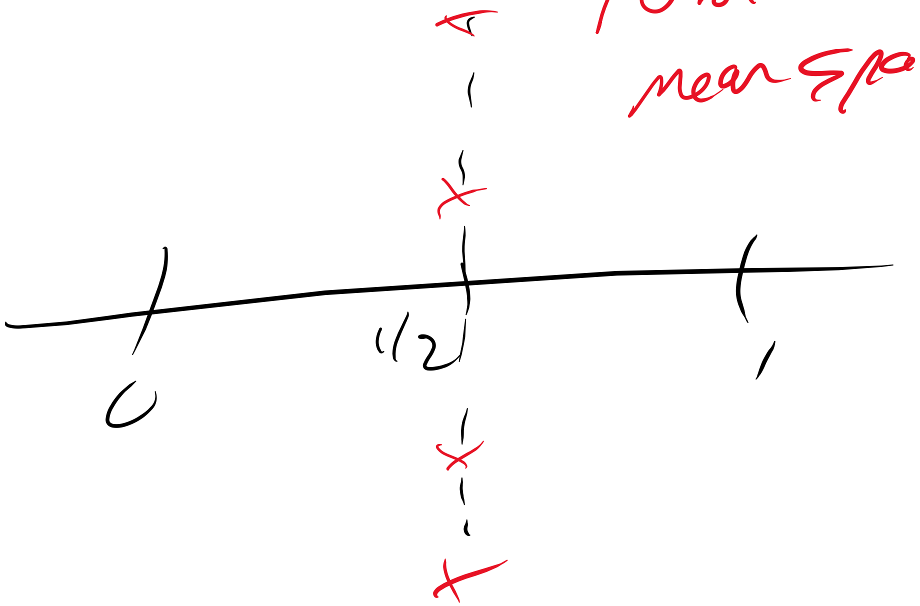
n-level corr



one L-fn is enough for averaging

n-level density

renormalize to mean spacing 1



need to average over similar L-functions

Similarities between L-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

nucleus

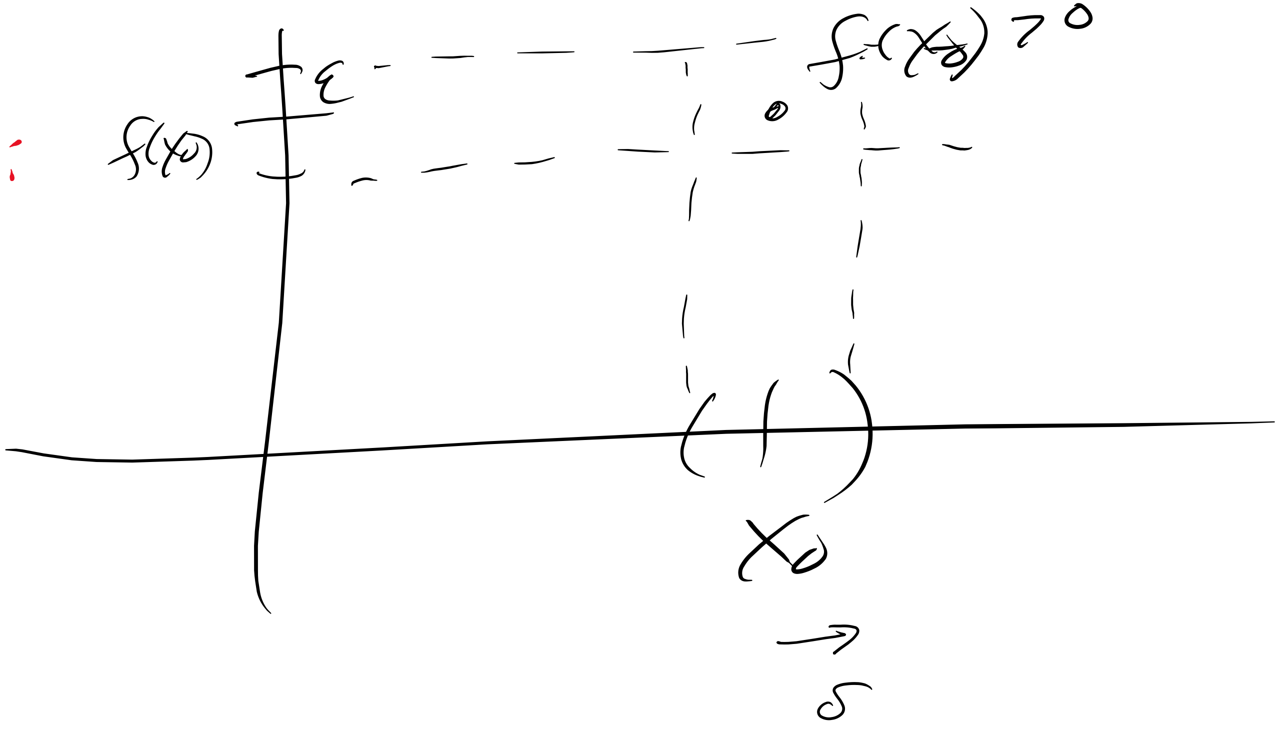
neutrons

physics

f and $\int_{-\infty}^{\infty} f(x)\, g(x)\, dx = 0$ for any cont, diff g

Assume f is Continuous: Then $f \equiv 0$

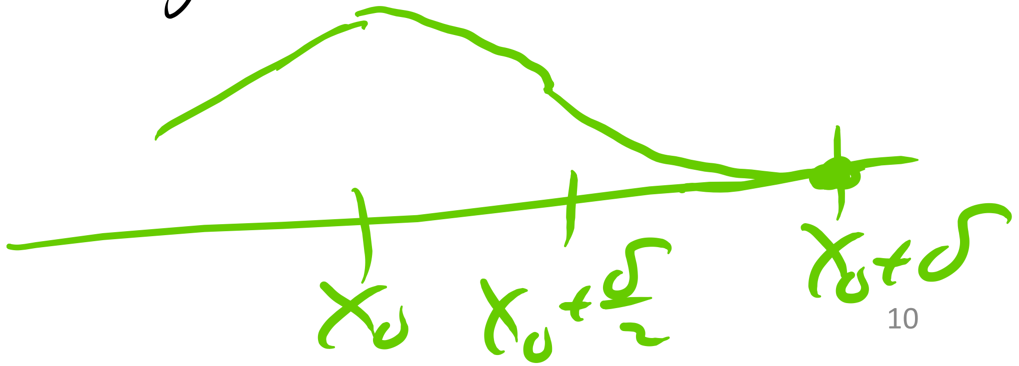
Proof:



$x_0 - \delta \le x \le x_0 + \delta$

here $f(x) > \frac{1}{2} f(x_0) > 0$

$g(x) =$



The more "g" we can study/use, the more we can say about f.

Imagine g is even

f and $\int_{-\infty}^{\infty} f(x)\, g(x)\, dx = 0$ for any cont, diff g

g even

Then f is ODD.

Exercise....