Linear Recurrences of Order at Most Two in Nontrivial Divisors

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Introduction

Definition 1

The set of small divisors of N is

$$S_N := \{d : 1 \leq d \leq \sqrt{N}, d \text{ divides } N\}.$$

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Definition 2

A positive integer N is said to be small recurrent if S'_N satisfies a linear recurrence of order at most two. When $|S'_N| \leq 2$, N is vacuously small recurrent.

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- Iannucci's key idea was to show that if S_N forms an AP, then the size $|S_N|$ cannot exceed 6. The trivial divisor 1 plays an important role in Iannucci's proofs.
- Recently, Chentouf generalized Iannucci's result from a different perspective by characterizing all N whose S_N satisfies a linear recurrence of order at most two.
- In particular, for each tuple $(u, v, a, b) \in \mathbb{Z}^4$, there is an integral linear recurrence, denoted by $U(u, v, a, b)$, of order at most two, given by

$$n_i = \begin{cases} u & \text{if } i = 1, \\ v & \text{if } i = 2, \\ an_{i-1} + bn_{i-2} & \text{if } i \geq 3. \end{cases}$$

- Noting that the appearance of the trivial divisor 1 contributes nontrivially to Chentouf's proof, we generalize his result: we characterize all positive integers N whose S'_N satisfies a linear recurrence of order at most two without the help of the trivial divisor.

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Proof: If all divisors (except 1) of N are divisible by p, then $N = p^k$ for some $k \geq 1$. Assume that N has a prime factor $q \neq p$. Then $q \geq \sqrt{N}$. Proposition 1 implies that N cannot have another prime factor at least $\sqrt{N}$. Hence, $N = p^k q$ for some $k \geq 1$ and $q > p^k$.

First case of S'_N

Notation

Write $S'_N = \{d_2, d_3, d_4, d_5, \ldots\}$. (We start with d_2 since the smallest divisor of N is usually denoted by $d_1 = 1$, which is excluded from S'_N.)

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If $d_3 = q$ for some prime $q > p$, then $d_4 = pq, p^2, r$ for some prime $r > q$.

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When $d_5 = r$, we show by induction the following proposition.

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Suppose that the first 4 numbers in S'_N are $p < q < p^2 < r$. Then $|S_N| \le 7$. As a result, $|S'_N| \le 6$.

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Hence, we can write

$$d_{2i-3} = (a^2 + b)\ell + a^k b^k s,$$

for some $\ell \in \mathbb{Z}$, some $k \geq 1$, and some $s \in \{p, q, p^2\}$.

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for some $\ell \in \mathbb{Z}$, some $k \geq 1$, and some $s \in \{p, q, p^2\}$. Since $d_{2i-3}|(a^2 + b)$, we have $d_{2i-3}|a^k b^k s$. By Lemma 3 iii), $d_{2i-3}|s$; that is, $d_{2i-3} \leq p^2$.

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for some $\ell \in \mathbb{Z}$, some $k \geq 1$, and some $s \in \{p, q, p^2\}$. Since $d_{2i-3} | (a^2 + b)$, we have $d_{2i-3} | a^k b^k s$. By Lemma 3 iii), $d_{2i-3} | s$; that is, $d_{2i-3} \leq p^2$. However, $d_{2i-3} \geq d_5 > d_4 = p^2$, a contradiction.

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Next we exclude other two cases by showing the following proposition.

Proposition 5

Suppose that the first 4 numbers in S'_N are $p < q < p^2 < r$. If N is small recurrent, then $|S'_N| \neq 4, 6$.

We conclude that $pd_{2i-3} \geq d_{2i+2}$. Since $pd_{2i-3} \in S'_N$, we know that $d_{2i+2} \in S'_N$ and $|S_N| \geq 2i+2$. This completes the proof to Proposition 4.

Next we exclude other two cases by showing the following proposition.

Proposition 5

Suppose that the first 4 numbers in S'_N are $p < q < p^2 < r$. If N is small recurrent, then $|S'_N| \neq 4, 6$.

Proof.

For each $N \in \mathbb{N}$ with the prime factorization $\prod_{i=1}^{\ell} p_i^{a_i}$, the divisor-counting function is

$$\tau(N) := \sum_{d|N} 1 = \prod_{i=1}^{\ell}(a_i + 1). \tag{1}$$

It is easy to verify that for $N > 1$,

$$\tau(N) := \begin{cases} 2|S'_N| + 3 & \text{if } N \text{ is a square,} \\ 2|S'_N| + 2 & \text{otherwise.} \end{cases} \tag{2}$$

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If $|S'_N| = 4$, then (2) gives $\tau(N) = 10$ or 11. Note that N has three distinct prime factors p, q, r and the power of p is at least 2. Since $2^3 \cdot 3 > 11$, N cannot have another prime factor besides p, q, r.

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Write $N = p^a q^b r^c$, for some $a \geq 2, b \geq 1, c \geq 1$. However, neither $(a+1)(b+1)(c+1) = 10$ nor $(a+1)(b+1)(c+1) = 11$ has a solution.

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Conclusion

By Propositions 4 and 5, we know that $|S'_N| = 5$; that is, $\tau(N) = 12$ or 13. Using the same reasoning as in the proof of Proposition 5, we know that $\tau(N) = 12$ and $N = p^2 qr$, where $p < q < p^2 < r$.

Full list

Now we give the complete list of small recurrent numbers.

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If N is small recurrent and $|S'_N| \geq 4$, then N belongs to one of the following forms.

(S1) $N = p^k$ or $N = p^k q$ for some $k \geq 1$ and $q > p^k$.

(S2) $N = pq^k$ or $pq^k r$ for some $k \geq 2$, $p < q$, and $pq^k < r$.

(S3) $N = p^k qr$ for some $k \geq 2$, some prime $r > p^k q$, and $\sqrt{q} < p < q$.

(S4) $N = p^k q$ for some $k \geq 2$ and $\sqrt{q} < p < q$.

(S5) $N = p^2 q^2$ for some $p < q < p^2$.

(S6) $N = p^3 q^2$ for some $p^{3/2} < q < p^2$.

(S7) $N = p^2 qr$, where the first four numbers in S'_N are $p < q < p^2 < r$.

We see the above list forms a necessary condition, so we now refine these forms to get a necssary and sufficient condition.

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Proposition 6

A positive integer N is small recurrent with $|S'_N| \geq 4$ if and only if N belongs to one of the following forms.

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Proposition 6

A positive integer N is small recurrent with $|S'_N| \geq 4$ if and only if N belongs to one of the following forms.

(S1) $N = p^k$ for some $k \geq 9$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^{\lfloor (k-1)/2 \rfloor}\}$ satisfies $U(p, p^2, p, 0)$.

(S2) $N = p^k q$ for some $k \geq 4$ and $q > p^k$. In this case, $S'_N = \{p, p^2, p^3, \ldots, p^k\}$ satisfies $U(p, p^2, p, 0)$.

(S3) $N = pq^k$ for some $k \geq 4$ and $p < q$. In this case, $S'_N = \{p, q, pq, q^2, \ldots\}$ satisfies $U(p, q, 0, q)$.

(S4) $N = pq^k r$ for some $k \geq 2$, $p < q$, and $r > pq^k$. In this case, $S'_N = \{p, q, pq, q^2, \ldots, pq^{k-1}, q^k, pq^k\}$ satisfies $U(p, q, 0, q)$.

(S5) $N = p^k q$ for some $k \geq 4$ and $\sqrt{q} < p < q$. In this case, $S'_N = \{p, q, p^2, pq, \ldots\}$ satisfies $U(p, q, 0, p)$.

(S6) $N = p^3q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.

(S7) $N = p^2qr$, where $p < q < p^2 < r < pq$, $(q^2 - p^3)|(pq - r)$, $(q^2 - p^3)|(rq - p^4)$, and $r = pq - \sqrt{(q^2 - p^3)(p^2 - q)}$. In this case, $S'_N = \{p, q, p^2, r, pq\}$ satisfies $U\left(p, q, \frac{p(pq-r)}{q^2-p^3}, \frac{rq-p^4}{q^2-p^3}\right)$.

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In our paper, we also characterize all the large recurrent numbers in a similar way. Here we just give the list:

(S6) $N = p^3q^2$ for some $p^{3/2} < q < p^2$. In this case, $S'_N = \{p, q, p^2, pq, p^3\}$ satisfies $U(p, q, 0, p)$.

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In our paper, we also characterize all the large recurrent numbers in a similar way. Here we just give the list:

Proposition 7

A number N is large recurrent with $|L'_N| \geq 4$ if and only if N belongs to one of the following forms.

(L1) $N = p^k$ for some $k \geq 9$. In this case, $L'_N = \{p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, \ldots, p^{k-1}\}$ satisfies $U(p^{\lceil (k-1)/2 \rceil + 1}, p^{\lceil (k-1)/2 \rceil + 2}, p, 0)$.

(L2) $N = p^kq$ for some $k \geq 4$ and $q > p^k$. In this case, $L'_N = \{q, pq, p^2q, \ldots, p^{k-1}q\}$ satisfies $U(q, pq, p, 0)$.

(L3) $N = p^k q$ for some $k \geq 4$ and $p^{k-1} < q < p^k$. Then

$$L'_N \;=\; \{p^k, pq, p^2q, \ldots, p^{k-1}q\}$$

satisfies $U(p^k, pq, p, 0)$.

(L4) $N = p^k q$ some for $k \geq 4$ and $p < q < p^2$. In this case,

$$L'_N \;=\; \begin{cases} \{p^{k/2+1}, p^{k/2}q, p^{k/2+2}, \ldots, p^{k-1}q\} & \text{if } 2|k, \\ \{p^{(k-1)/2}q, p^{(k+3)/2}, p^{(k+1)/2}q, \ldots, p^{k-1}q\} & \text{if } 2 \nmid k. \end{cases}$$

Observe that L'_N satisfies $U(p^{k/2+1}, p^{k/2}q, 0, p)$ and $U(p^{(k-1)/2}q, p^{(k+3)/2}, 0, p)$ for even and odd k, respectively.

(L5) $N = p^4 q$ with $p^2 < q < p^3$, $(p^5 - q^2)|(p^2 - q)$, and $(p^5 - q^2)|(p^3 - q)$. In this case, $L'_N \;=\; \{pq, p^4, p^2q, p^3q\}$.

(L6) $N = p^3q^2$ for $p < q < p^2$. In this case, $L'_N = \{q^2, p^2q, pq^2, p^3q, p^2q^2\}$ satisfies $U(q^2, p^2q, 0, p)$.

(L7) $N = pq^k$ for some $k \geq 4$ and $p < q$. In this case,

$$L'_N = \begin{cases} \{pq^{\frac{k}{2}}, q^{\frac{k}{2}+1}, \dots, q^k\} & \text{if } 2|k, \\ \{q^{\frac{k+1}{2}}, pq^{\frac{k+1}{2}}, \dots, q^k\} & \text{if } 2 \nmid k. \end{cases}$$

Observe that L'_N satisfies $U(pq^{k/2}, q^{k/2+1}, 0, q)$ and $U(q^{(k+1)/2}, pq^{(k+1)/2}, 0, q)$ for even and odd k, respectively.

(L8) $N = pq^k r$ for some $k \geq 2$ and $p < q < pq^k < r$. In this case, $L'_N = \{r, pr, qr, pqr, q^2 r, \dots, q^k r\}$ satisfies $U(r, pr, 0, q)$.

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