Distribution of Missing Sums in Correlated Sumsets

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Introduction

Given $A \subseteq \{0, \ldots, n-1\}$, with $|A|$ its size, define its sumset

- $A + A = \{a_1 + a_2 \mid a_1, a_2 \in A\} \subseteq \{0, \ldots, 2n-2\}$.

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- $A + A = \{a_1 + a_2 \mid a_1, a_2 \in A\} \subseteq \{0, \ldots, 2n-2\}$.
- Recent research in $|A + A|$ as a random variable
- Set $\mathbb{P}(i \in A) = p$, where $p \in [0, 1]$ and $q := 1 - p$.
- Martin and O'Bryant's formative paper [MO] compared $|A + A|$ to $|A - A|$.

Motivating Questions

- What is $\mathbb{E}[|A + A|]$?
- What is $\text{Var}(|A + A|)$?

Prior Work

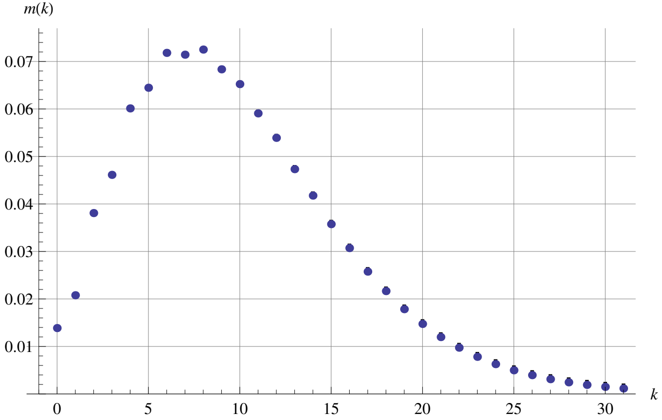
Theorem (Martin and O'Bryant '06)

If $p = \frac{1}{2}$, *then* $\mathbb{E}[|A + A|] = 2n - 1 - 10 + O((3/4)^{n/2})$.

- Remember $A \subset \{0, \dots, n-1\}$ gives $A + A \subset \{0, \dots, 2n-2\}$.

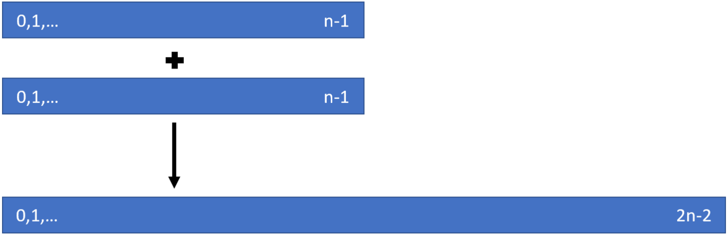
Numerical Experimentation

[LMO] plotted the frequency with which k sums are missing from $A + A$.



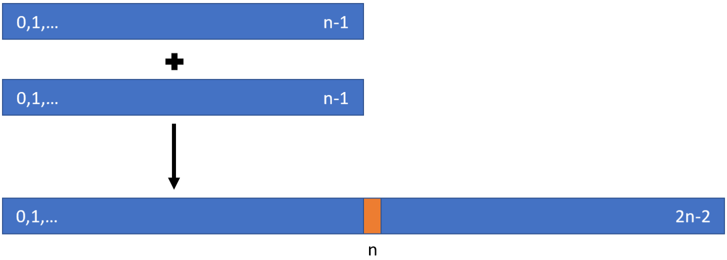
Problem Intuition

We are adding elements of $0, \ldots, n-1$ to get elements of $0, \ldots, 2n-2$; we need to estimate how many we will find



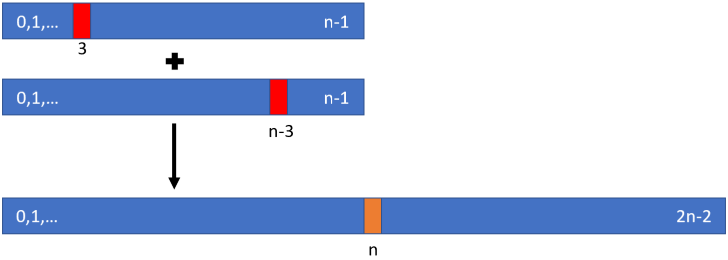
Problem Intuition

How likely is it that $n \in A + A$?



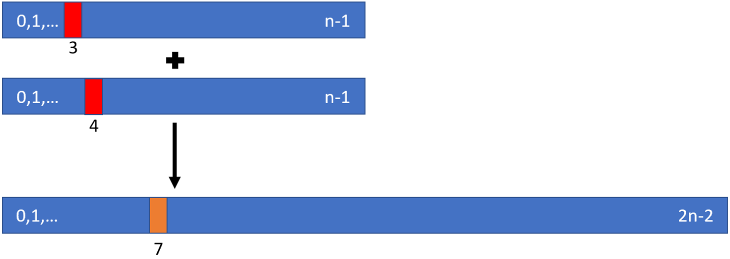
Problem Intuition

There are $O(n)$ distinct pairs which, if present, sum to n.



Problem Intuition

For smaller numbers, there are fewer pairs. For example, 7 has only $0+7, 1+6, 2+5, 3+4$, not matter the value of $n > 7$.



Problem Intuition

The takeaway: numbers in the middle of $0, \ldots, 2n-2$ are likely to appear, while numbers at the upper and lower extremes are more likely to be missing.

Prior Work

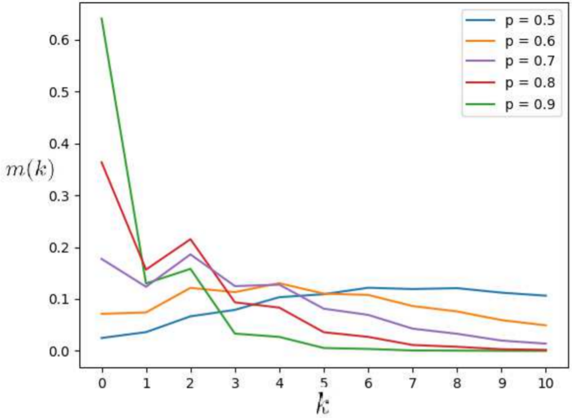
Theorem (Martin and O'Bryant '06)

If $p = \frac{1}{2}$, *then* $\mathbb{E}[|A + A|] = 2n - 1 - 10 + O((3/4)^{n/2})$.

- Can we compute the same for generic p?
- Problem: not all sets A are equally likely...

Numerics

We can plot still try numerics and investigate the frequency with which k sums are missing from $A+A$, for different $p \in [0,1]$.



Results

Theorem (CHKLMSX)

For $p \in [0,1]$ and $q := 1-p$,

$$\mathbb{E}[|A+A|] = \sum_{r=0}^{n} p^r q^{n-r} \binom{n}{r} \left(2 \sum_{k=0}^{n-1} \left(1 - \frac{f(k)}{\binom{n}{r}} \right) - \left(1 - \frac{f(n-1)}{\binom{n}{r}} \right) \right),$$

where

$$f(k) = \begin{cases} \sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i} \binom{\frac{k+1}{2}}{i-\frac{k+1}{2}} \binom{n-k-1}{r-i} & \text{for } k \text{ odd} \\ \sum_{i=\frac{k}{2}}^{k} 2^{k-i} \binom{\frac{k}{2}}{i-\frac{k}{2}} \binom{n-k-1}{r-1-i} & \text{for } k \text{ even}. \end{cases}$$

In particular, where the LHS holds for $p > \frac{1}{2}$

$$2n-1-2q\,\frac{1}{1-\sqrt{2q}} - (2q)^{\frac{n-1}{2}} \le \mathbb{E}[|A+A|] \le 2n-1-2q\,\frac{1-q^{\frac{n-1}{2}}}{1-\sqrt{q}}$$

Reducing Expected Value

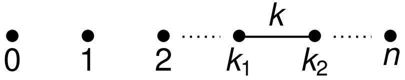
For **any** p, all subsets of $\{0, ..., n-1\}$ with equal cardinality have the same probability of occurring.

$$\mathbb{E}[|A+A|] = \sum_{r=0}^{n} \mathbb{P}(|A| = r) \sum_{k=0}^{2n-2} \mathbb{P}(k \in A+A \mid |A| = r)$$
$$= \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} \sum_{k=0}^{2n-2} \left(1 - \mathbb{P}(k \notin A+A \mid |A| = r)\right)$$

Our target: $\mathbb{P}(k \notin A+A \mid |A| = r)$

A Graph Theoretic Solution

- $G = (V, E)$, $V = \{0, \ldots, n-1\}$
- Edge (k_1, k_2) if $k_1 + k_2 = k$
- A corresponds to a subset of these vertices
- A vertex cover of missing elements corresponds to $k \notin A + A$
- This graph is a collection of disjoint edges and isolated vertices



Vertex Cover Definition

Vertex Cover Definition

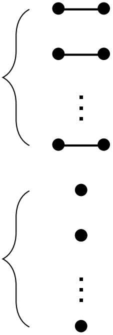
A vertex cover V' of an undirected graph $G = (V, E)$ is a subset of V such that $uv \in E \Rightarrow u \in V' \vee v \in V'$

A Graph Theoretic Solution

- Since $|A| = r$, we are looking for the number of $n - r$-vertex vertex covers

ℓ disjoint edges

$n - 2\ell$ isolated vertices



Computing $\mathbb{E}[|A+A|]$

Lemma (CHKLMSX)

$$\mathbb{P}[k \notin A+A \mid |A|=r] = \begin{cases} \dfrac{\sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i}\binom{\frac{k+1}{2}}{i-\frac{k+1}{2}}\binom{n-k-1}{r-i}}{\binom{n}{r}} & \text{for } k \text{ odd} \\ \dfrac{\sum_{i=\frac{k}{2}}^{k} 2^{k-i}\binom{\frac{k}{2}}{i-\frac{k}{2}}\binom{n-k-1}{r-1-i}}{\binom{n}{r}} & \text{for } k \text{ even} \end{cases}$$

Implementing Combinatorics

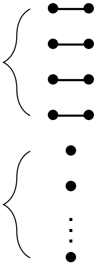
$$\mathbb{P}[k \notin A + A \mid |A| = n - r] = \frac{\text{ways to place r vertices and get cover}}{\text{ways to choose r vertices from n}}$$

$$= \frac{\sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i} \binom{\frac{k+1}{2}}{i-\frac{k+1}{2}} \binom{n-k-1}{r-i}}{\binom{n}{r}}$$

$$r = r - i + i$$
$$= r - i + \frac{k+1}{2} + i - \frac{k+1}{2}$$

$\frac{k+1}{2}$ disjoint edges

$n - k - 1$ isolated vertices



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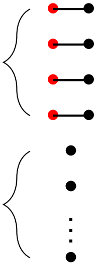

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$$\begin{aligned} r &= r - i + i \\ &= r - i + \frac{k+1}{2} + i - \frac{k+1}{2} \end{aligned}$$

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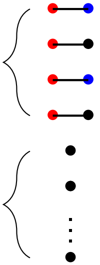

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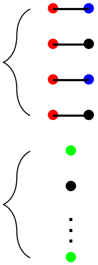

Variance: A Problem with Dependencies

- $\mathbb{P}[i \notin A + A]$ is straightforward.

Lemma (Martin and O'Bryant '06)

Let $q = 1 - p$. If $i \leq n - 1$,

$$\mathbb{P}[i \notin A + A] = \begin{cases} (2q - q^2)^{(i+1)/2} & \text{for } i \text{ odd} \\ q(2q - q^2)^{i/2} & \text{for } i \text{ even} \end{cases}$$

- However, $\mathbb{P}[i, j \notin A + A]$, required to compute Variance, is laden with dependencies
- Example: $\mathbb{P}[0 \notin A + A] = 1 - p$, $\mathbb{P}[1 \notin A + A] = 1 - p^2$, but $\mathbb{P}[0, 1 \notin A + A] = 1 - p^2$
- Analyzed for the $p = \frac{1}{2}$ case in [LMO]
- We work on the problem for generic p

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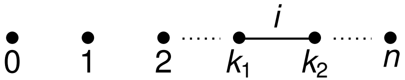
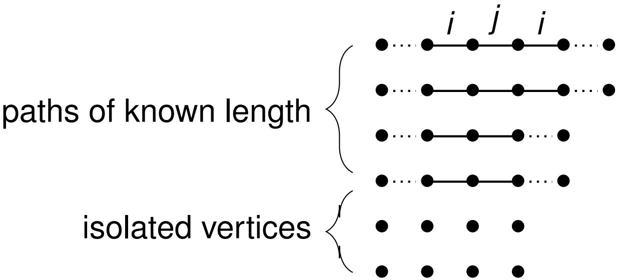


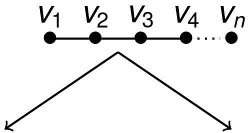
Structure of the Graph

- This graph is the union of disjoint paths [LMO]
- Understanding vertex covers reduces to understanding vertex covers of paths



Prior Work - Fibonacci Numbers

- When $p = \frac{1}{2}$, all sets equally likely, [LMO] only needed to count vertex covers
- How can we count vertex covers on a path of length n?



Case 1: $1 \in S$
12 edge is covered

Case 2: $1 \notin S$, then necessarily $2 \in S$, 23 edge is covered

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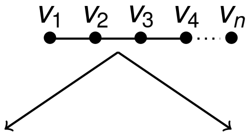

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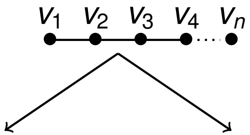
- $F_n := \#$ of vertex covers
- $F_n = F_{n-1} + F_{n-2}$; Fibonacci!

Vertex Cover Probabilities

- How can we compute the probability of finding a vertex cover on a path of length n?



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Case 2: $1 \notin S$, then necessarily $2 \in S$, 23 edge is covered

- $a_n := \mathbb{P}(\text{ a vertex cover })$
- $a_n = qa_{n-1} + pqa_{n-2}$; a recurrence relation we can solve

Vertex Cover Probabilities

Lemma

Set $\phi(p) := \sqrt{1 + 2p - 3p^2}$*. Then*

$$a_n = \frac{(\phi(p) - 1 - p))(1 - p - \phi(p))^n + (\phi(p) + 1 + p)(1 - p + \phi(p))^n}{2^{n+1}\phi(p)}$$

This can be used to compute the variance of $|A + A|$.

A Generalization to Correlated Sumsets

- Introduced by 2013 SMALL REU group [DKMMW]
- Replace $A + A$ with $A + B$, where
  - $\mathbb{P}(i \in A) = p$
  - $\mathbb{P}(i \in B \mid i \in A) = p_1$
  - $\mathbb{P}(i \in B \mid i \notin A) = p_2$
- $p_1 = 1, p_2 = 0$ reduces to $A + A$
- Once again, determining $\mathbb{P}(i, j \notin A + B)$ is difficult

Generalizing the Graph Framework

- $G = (V, E)$, $V = \{0_A, \ldots, (n-1)_A, 0_B, \ldots, (n-1)_B\}$
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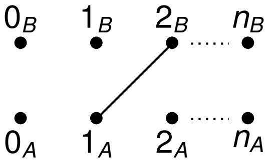

Future Work

- Find a lower bound on $\mathbb{E}[|A+A|]$ for $p \leq \frac{1}{2}$.
- Find and analyze a closed form for a_n in the correlated sets case.
- Find $\mathbb{E}[|A+B|]$ and $\mathrm{Var}(|A+B|)$ for any correlated sumset $A+B$.

Bibliography

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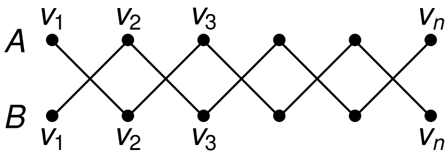
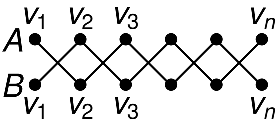


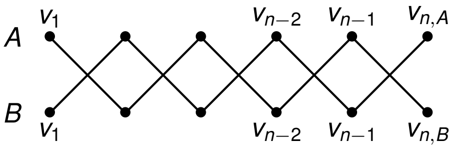
Correlated Set Recurrence Relation

- How can we compute the probability of finding a vertex cover on a pair of paths of length n?



- Many cases, based on whether we have $1 \in A$ and/or $1 \in B$
- Solution: system of recurrence relations

Correlated Set Recurrence Relation



$$a_n := \mathbb{P}(\text{ a vertex cover })$$
$$b_n := \mathbb{P}(\text{ a vertex cover AND } n_A \in A)$$
$$c_n := \mathbb{P}(\text{ a vertex cover AND } n_B \in B)$$

then we find that

$$a_n = qq_2 a_{n-1} + qp_2 b_{n-1} + pq_1 c_{n-1} + pp_1 qq_2 a_{n-2}$$
$$b_n = qq_2 a_{n-1} + qp_2 b_{n-1}$$
$$c_n = qq_2 a_{n-1} + pq_1 c_{n-1}$$