Left and Right Quotient Sets in Non-Abelian Groups

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Background

Definitions

The famous More-Sums-Than-Differences (MSTD) problem asks if $|A + A| > |A - A|$ for $A \subseteq \mathbb{Z}$, where $A + A$ denotes the set of all distinct sums and $A - A$ is the set of all distinct differences formed by the elements of A.

Definitions

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In 2006, Martin and O'Bryant proved that a positive percentage of sets are sum-dominant. [MO06]

Related problem

What if A is non-commutative?
$AA^{-1} \neq A^{-1}A$.

Definitions

Definition

Right Quotient Set:

- $AA^{-1} := \{a_i \cdot a_j^{-1} : a_i, a_j \in A\}$

Left Quotient Set:

- $A^{-1}A := \{a_i^{-1} \cdot a_j : a_i, a_j \in A\}$

Free groups

Definition

Let $A = \{x, y\}$ be generators. $F(A) = F_2$ is the group of **words** on $\{x, y, x^{-1}, y^{-1}\}$.

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$$x * x^2 = x^3$$
$$x * x^{-1} = e$$
$$x * y = xy.$$

Graph of F_2

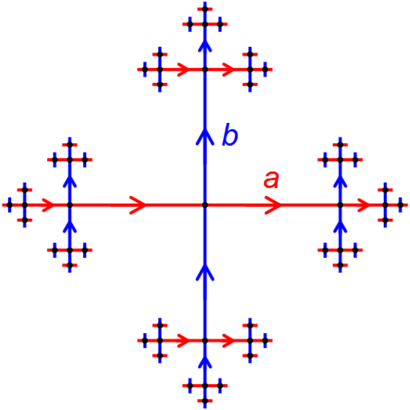

Figure: Visualization of group multiplication in F_2

Example (Quotient set example)

Example

	x	y
y^{-1}		
x^{-1}		

Table: “Multiplication table” of $A = \{x, y\}$ with $A^{-1} = \{x^{-1}, y^{-1}\}$

Definitions

Definition

A set $A \subseteq G$ is either

- **More Right Then Left (MRTL)**: $|AA^{-1}| > |A^{-1}A|$.
- **Balanced**: $|AA^{-1}| = |A^{-1}A|$.
- **More Left Then Right (MLTR)**: $|A^{-1}A| > |AA^{-1}|$.

Motivation

Question: What are the possible values of $|AA^{-1}| - |A^{-1}A|$ for finite subsets $A \subseteq G$?

Example

If G is abelian, then $AA^{-1} = A^{-1}A$. Thus, the only possible value of $|AA^{-1}| - |A^{-1}A|$ is 0.

History

Tao remarks

> *there is no relation between the size of AA^{-1} and $A^{-1}A$ in general. For instance, if H is a multiplicative set which is also a subgroup of G, and $A := (x \cdot H) \cup H$ for some x not in the normaliser of H, then AA^{-1} has about the same size as H, but $A^{-1}A$ can be much larger. [Tao11]*

The Difference Graph

Motivating example

	x	y
y^{-1}	xy^{-1}	e
x^{-1}	e	yx^{-1}

$$\xrightarrow{\text{graph structure}}$$

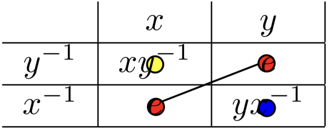

Figure: Associated to the "multiplication table" of $\{x, y\}$ is a graph.

Definition

Definition

Given a finite subset $A \subseteq G$ with $|A| = n$, the **difference graph** $D_A = (V, E)$ is defined as follows.

- Vertex set is $V := [n] \times [n]$
- Edge set is $E(D_A) := [(i, j), (k, \ell)] \iff a_i a_j^{-1} = a_k a_\ell^{-1}$

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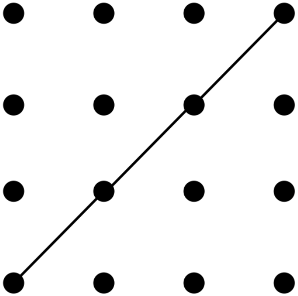
Let $\mathcal{C}(D_A)$ be the set of connected components of D_A and $c(D_A) = |\mathcal{C}(D_A)|$ be the number of connected components.

Note that $c(D_A) = |AA^{-1}|$.

Properties of D_A

$$\begin{array}{c|cccc}
 & a_1 & a_2 & a_3 & a_4 \\
a_4^{-1} & \bullet & \bullet & \bullet & \bullet \\
a_3^{-1} & \bullet & \bullet & \bullet & \bullet \\
a_2^{-1} & \bullet & \bullet & \bullet & \bullet \\
a_1^{-1} & \bullet & \bullet & \bullet & \bullet
\end{array}$$

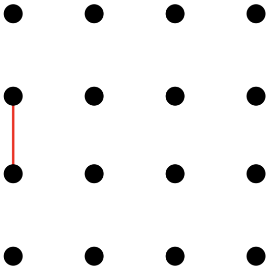
Properties of D_A



$$\longleftrightarrow \quad a_1 a_1^{-1} = a_2 a_2^{-1} = e$$

Figure: Diagonal edges are always present.

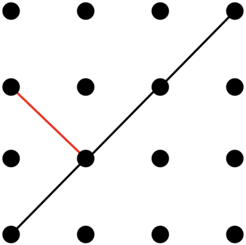
Properties of D_A



$$\longleftrightarrow a_1 a_2^{-1} = a_1 a_3^{-1} \implies a_2 = a_3$$

Figure: Any edge parallel to the x or y axis is not present.

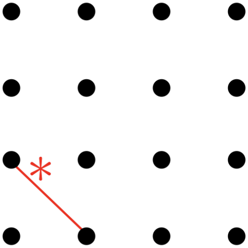
Properties of D_A



$$\longleftrightarrow \quad a_1 a_3^{-1} = a_2 a_2^{-1} = e \implies a_1 = a_3$$

Figure: No vertex connects to the diagonal.

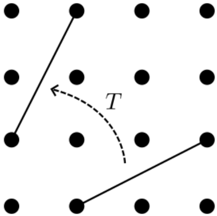
Properties of D_A



$$\longleftrightarrow \quad a_1 a_2^{-1} = a_2 a_1^{-1} \implies (a_1 a_2^{-1})^2 = e$$

Figure: **If** G **has no elements of order** 2, no vertex connects to its "transpose"

Properties of D_A



$$\longleftrightarrow \quad \begin{aligned} & a_2a_1^{-1} = a_4a_2^{-1} \\ & \implies a_1a_2^{-1} = (a_2a_1^{-1})^{-1} = (a_4a_2^{-1})^{-1} = a_2a_4^{-1} \end{aligned}$$

Figure: If any edge is present, its “transpose” is also present.

Properties of D_A

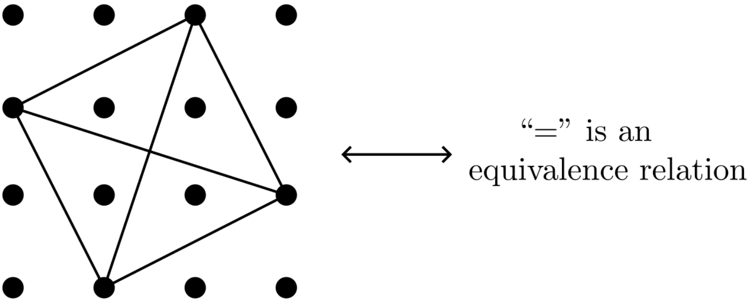


Figure: Connected components form a clique.

Summary of properties

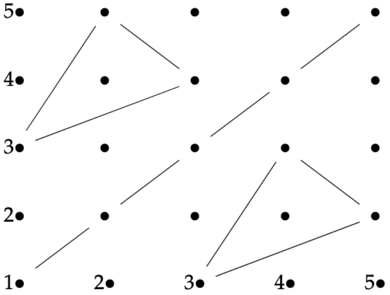
Lemma

Let $i, j, k, \ell \in [n]$.

1. *The following edges are forbidden in* $E(D_A)$*:*
    1. $[(i,j),(k,k)]$*: an edge connecting to the diagonal, provided that* $i \neq j$*.*
    2. $[(i,j),(i,k)]$ *(or* $[(j,i),(k,i)]$*, but this is the handled by (1)): an edge connecting vertices on the same axis.*
    3. ***If*** G ***has no elements of order*** 2*,* $[(i,j),(j,i)]$*: an edge connecting to its symmetric pair, provided that* $j \neq i$*.*
2. $[(i,i),(j,j)] \in E(D_A)$.
3. $[(i,j),(k,\ell)] \in E(D_A) \iff [(j,i),(\ell,k)] \in E(D_A)$.
4. *If* C *is a connected component in* D_A*, then* C *is a clique.*

Example

For $n = 5$, $A = \{x^2, (xy)^{-1}, (xy)^{-1}x(xy)^{-1}, x(xy^{-1}), (xy)^{-1}(xy^{-1})\} \subseteq F_2$ has the following difference graph:



Bijection of Edges

Using the fact:

$$a_i a_j^{-1} = a_k a_\ell^{-1} \iff a_k^{-1} a_i = a_\ell^{-1} a_j$$

We can obtain a **bijection of edges**

$$\phi\colon E(D_A) \to E(D_{A^{-1}})$$
$$[(i,j),(k,\ell)] \mapsto [(k,i),(\ell,j)].$$

Bijection of Edges

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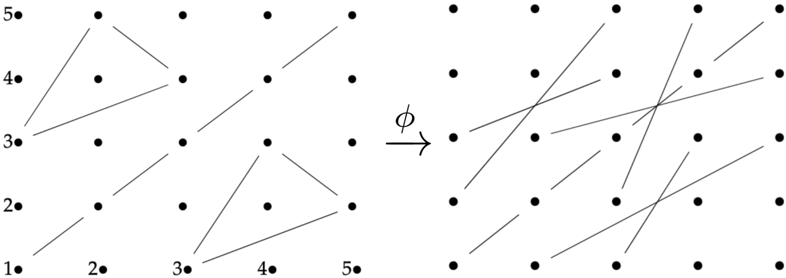
$$[(i,j),(k,\ell)] \mapsto [(k,i),(\ell,j)].$$

Remark

*Tao uses this to prove the **additive energies** $\Lambda(A, A^{-1})$ and $\Lambda(A^{-1}, A)$ are equal.*

Bijection of Edges Example



17 components 15 components

Figure: ϕ reduces the number of connected components

Nonzero Cardinality Difference

Nonzero Cardinality Difference

Theorem (SMALL 2025)

Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

- Casework: $|A| = 1, 2, 3$.

No Elements of Order 2

Definition

Recall that a group G is said to have **no elements of order 2** if for every element $g \in G$ with $g \neq e$ we have $g^2 \neq e$.

An example is $\mathbb{Z}/3\mathbb{Z}$.

No Elements of Order 2

Theorem

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

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Proof sketch.

We prove a stronger result: $|AA^{-1}|$ is always odd.
A connected component is either (1) symmetric under transposition, or (2) is disjoint from its transpose. Only one component is in the first class: the diagonal. All other components come in pairs.

$$\underbrace{\text{odd}}_{(1)} + \underbrace{\text{even}}_{(2)} = \text{odd}. \qquad \square$$

No Elements of Order 2

Theorem

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq \blacksquare$.

Guesses?

Hint: Without restriction, $|A| \geq 4$ (previous theorem).

No Elements of Order 2

Theorem

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 5$.

Lemma

Let $|A| = n$. Then D_A has no connected component (other then the diagonal) with more than n elements.

- By Contradiction.

Continuation

Lemma

Suppose $|A| = 4$. If group G does not have an element of order 2 and its largest possible cycle is K_4, then the number of connected components in D_A is equal to the number of connected components in $D_{A^{-1}}$.

- The difference graph is heavily used to this lemma.
- We do casework on triangles.
- ϕ leaves the number of connected components unchanged.

The Free Group on 2 Generators

The Free Group on 2 Generators

Theorem

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = 2n$.

The Free Group on 2 Generators

Theorem

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n$.

The following set in F_3 with $n = 1$:

$$A := \{x, y^{-1}, y^{-1}xy^{-1}, xz, y^{-1}z\}$$

has

$$|AA^{-1}| - |A^{-1}A| = 2.$$

The Free Group on 2 Generators

Theorem

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n$.

More generally for $n \geq 1$, A_n is constructed as a subset of $F_{3n} = F(\{x_1, y_1, z_1, \ldots, x_n, y_n, z_n\})$ as follows:

$$A_n := \bigcup_{i=1}^{n} \{x_i, y_i^{-1}, y_i^{-1}x_iy_i^{-1}, x_iz_i, y_i^{-1}z_i\}$$

We prove

$$|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n.$$

The Infinite Dihedral Group

Definition

Definition

The **Infinite Dihedral group**, denoted D_∞ is defined as followed: $D_\infty := \langle r, s | s^2 = e, srs = r^{-1} \rangle$

- r: a generator representing translation
- s: a generator representing reflection

Key Properties:

- Non-abelian: $rs \neq sr$
- Has elements of order 2

The Infinite Dihedral Group

Theorem

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = n$

The Infinite Dihedral Group

Theorem

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = n$

Proof sketch.

D_∞ has two copies of $\mathbb{Z}$: $\langle r \rangle$ and $s\langle r \rangle$.

The Infinite Dihedral Group

Theorem

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = n$

Proof sketch.

D_∞ has two copies of $\mathbb{Z}$: $\langle r \rangle$ and $s\langle r \rangle$. Let $B \subseteq \mathbb{Z}$ be finite and let $A = \{r^b, sr^b : b \in B\}$. Then

$$|AA^{-1}| - |A^{-1}A| = |B - B| - |B + B|.$$

A result of Martin and O'Bryant says this ranges over all $n \in \mathbb{Z}$. [MO06]. $\square$

Summary of Results

Results

Theorem (SMALL 2025)

Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

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Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

This is sharp: the quasidihedral group of order 16 has a subset of size 4 satisfying the above theorem.

Results

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 5$.

Results on possible values

Question: possible values of $|AA^{-1}| - |A^{-1}A|$?

Theorem

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

Acknowledgements

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