More Sums than Differences with a Quotient Perspective

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Background

Background

Definition

For a set $A \subseteq \mathbb{Z}$,

$$A + A := \{a_i + a_j : a_i, a_j \in A\},$$

$$A - A := \{a_i - a_j : a_i, a_j \in A\}.$$

The famous More-Sums-Than-Differences (MSTD) problem asks if $|A + A| > |A - A|$ for finite subsets $A \subseteq \mathbb{Z}$. In 2006, Martin and O'Bryant proved that a positive percentage of sets are sum-dominant [MO06].

A Related Problem

Definition

Right Quotient Set:

- $AA^{-1} := \{a_i \cdot a_j^{-1} : a_i, a_j \in A\}$

Left Quotient Set:

- $A^{-1}A := \{a_i^{-1} \cdot a_j : a_i, a_j \in A\}$

If A is non-commutative, $AA^{-1} \neq A^{-1}A$.

Free groups

Definition

Let $A = \{x, y\}$ be generators. $F(A) = F_2$ is the group of **words** on $\{x, y, x^{-1}, y^{-1}\}$.

Example

For example,

$$x * x^2 = x^3$$
$$x * x^{-1} = e$$
$$x * y = xy.$$

Multiplication in Free Groups

Example

	x	y
y^{-1}	xy^{-1}	e
x^{-1}	e	yx^{-1}

Table: "Multiplication table" in F_2 of AA^{-1} with $A = \{x, y\}$.

Motivation

Question: For G a group, what are the possible values of $|AA^{-1}| - |A^{-1}A|$ for finite subsets $A \subseteq G$?

Example

If G is abelian, then $AA^{-1} = A^{-1}A$ for every $A \subseteq G$, so the only possible value of $|AA^{-1}| - |A^{-1}A|$ is 0.

The Difference Graph

Motivating example

	x	y
y^{-1}	xy^{-1}	e
x^{-1}	e	yx^{-1}

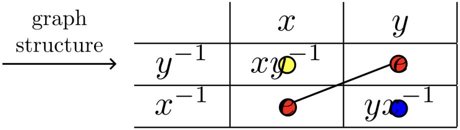


Figure: Associated to the "multiplication table" in F_2 for AA^{-1} with $A = \{x, y\}$ is a graph.

Definition

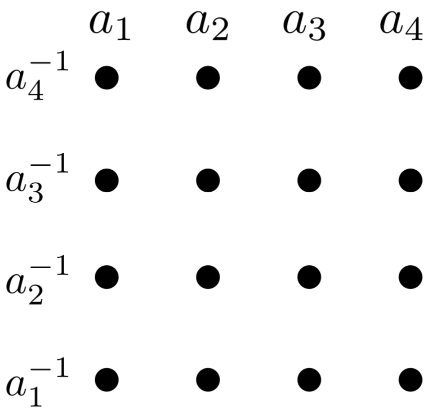
Definition

Given a finite subset $A \subseteq G$ with $|A| = n$, the **difference graph** $D_A = (V, E)$ is given by

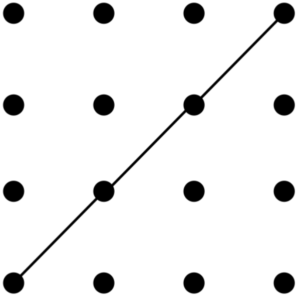
- Vertex set $V := [n] \times [n]$
- Edge set $E(D_A) := [(i, j), (k, \ell)] \iff a_i a_j^{-1} = a_k a_\ell^{-1}$

The number of connected components in D_A is equal to $|AA^{-1}|$.

Properties of D_A



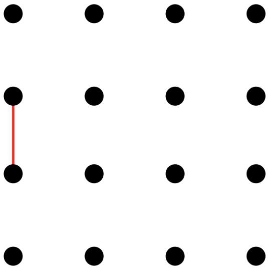
Properties of D_A



$$\longleftrightarrow \quad a_1 a_1^{-1} = a_2 a_2^{-1} = e$$

Figure: Diagonal edges are always present.

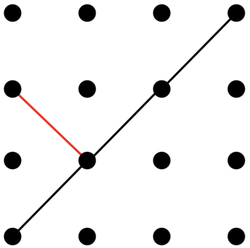
Properties of D_A



$$\longleftrightarrow a_1 a_2^{-1} = a_1 a_3^{-1} \implies a_2 = a_3$$

Figure: Horizontal and vertical edges are not present.

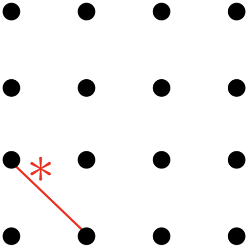
Properties of D_A



$$\longleftrightarrow \quad a_1 a_3^{-1} = a_2 a_2^{-1} = e \implies a_1 = a_3$$

Figure: No vertex not on the diagonal shares an edge with a vertex on the diagonal.

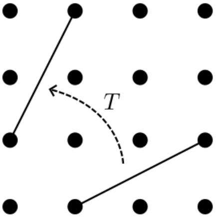
Properties of D_A



$$\longleftrightarrow \quad a_1 a_2^{-1} = a_2 a_1^{-1} \implies (a_1 a_2^{-1})^2 = e$$

Figure: If G has no elements of order 2, no vertex connects to its "transpose".

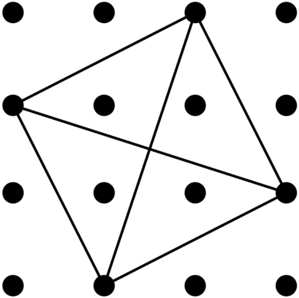
Properties of D_A



$$\longleftrightarrow \quad a_2a_1^{-1} = a_4a_2^{-1}$$
$$\implies a_1a_2^{-1} = (a_2a_1^{-1})^{-1} = (a_4a_2^{-1})^{-1} = a_2a_4^{-1}$$

Figure: If any edge is present, its “transpose” is also present.

Properties of D_A

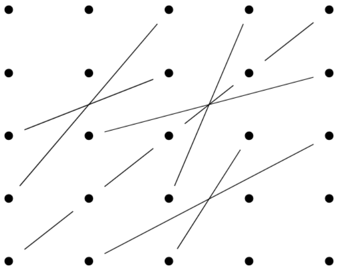


$\longleftrightarrow$ “=” is an equivalence relation

Figure: Connected components form a clique.

Example

For $n = 5$,
$A = \{x^2, x^3y^{-1}, y^{-1}x^{-1}, y^{-1}x^{-3}y^{-1}x^{-1}, y^{-2}\} \subseteq F_2$ has the following difference graph:



Bijection of Edges

Using the fact:

$$a_i a_j^{-1} = a_k a_\ell^{-1} \iff a_k^{-1} a_i = a_\ell^{-1} a_j$$

We can obtain a **bijection of edges**

$$\phi\colon E(D_A) \to E(D_{A^{-1}})$$
$$[(i,j),(k,\ell)] \mapsto [(k,i),(\ell,j)].$$

Bijection of Edges Example

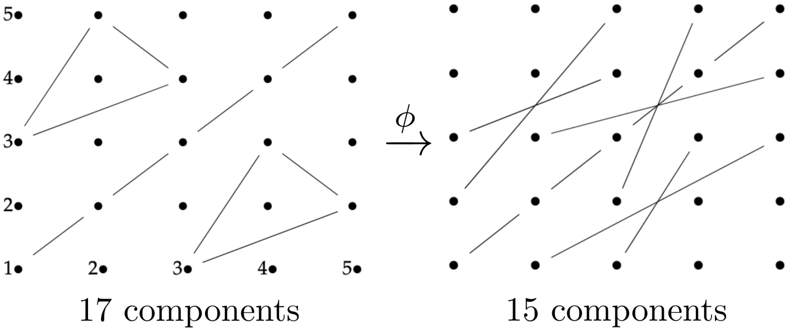


Figure: ϕ reduces the number of connected components.

Nonzero Cardinality Difference

Nonzero Cardinality Difference

Theorem (SMALL 2025)

Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

This is sharp: the quasidihedral group of order 16 has a subset of size 4 satisfying the above theorem.

No Elements of Order 2

No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

Proof sketch.

We prove a stronger result: $|AA^{-1}|$ is always odd.
A connected component is either (1) symmetric under transposition, or (2) is disjoint from its transpose. Only the diagonal is in the first class. All other components come in pairs.

$$\underbrace{\text{odd}}_{(1)} + \underbrace{\text{even}}_{(2)} = \text{odd}.$$

□

No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 5$.

Continuation

In order to prove this, we utilize the following lemmas.

Lemma (SMALL 2025)

Let $|A| = n$. Then D_A has no connected component (other then the diagonal) with more than n elements.

Lemma (SMALL 2025)

Suppose $|A| = 4$. If group G does not have an element of order 2 and its largest possible cycle is C_4, then the number of connected components in D_A is equal to the number of connected components in $D_{A^{-1}}$.

- We do casework on triangles.
- ϕ leaves the number of connected components unchanged.

The Free Group on 2 Generators

The Free Group on 2 Generators

Theorem (SMALL 2025)

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n$.

For $n = 1$, $A \subseteq F_3$ given by

$$A := \{x, y^{-1}, y^{-1}x^{-1}y^{-1}, xz, y^{-1}z\}$$

has

$$|AA^{-1}| - |A^{-1}A| = 2.$$

The Free Group on 2 Generators

Theorem (SMALL 2025)

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n$.

More generally, for $n \geq 1$, A_n is constructed as a subset of $F_{3n} = F(\{x_1, y_1, z_1, \ldots, x_n, y_n, z_n\})$ as follows:

$$A_n := \bigcup_{i=1}^{n} \{x_i, y_i^{-1}, y_i^{-1}x_i^{-1}y_i^{-1}, x_iz_i, y_i^{-1}z_i\}$$

We prove

$$|A_nA_n^{-1}| - |A_n^{-1}A_n| = 2n.$$

The Infinite Dihedral Group

Definition

Definition

The **Infinite Dihedral group** is given by

$$D_\infty := \langle r, s | s^2 = e, srs = r^{-1} \rangle.$$

- r: a generator representing translation
- s: a generator representing reflection

- Has elements of order 2

The Infinite Dihedral Group

Theorem (SMALL 2025)

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that $|A_nA_n^{-1}| - |A_n^{-1}A_n| = n$.

The Infinite Dihedral Group

Theorem (SMALL 2025)

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = n$.

Proof sketch.

D_∞ has two copies of $\mathbb{Z}$: $\langle r \rangle$ and $s\langle r \rangle$. Let $B \subseteq \mathbb{Z}$ be finite and let $A = \{r^b, sr^b : b \in B\}$. Then

$$|AA^{-1}| - |A^{-1}A| = |B - B| - |B + B|.$$

A result of Martin and O'Bryant says this ranges over all $n \in \mathbb{Z}$ [MO06]. □

Acknowledgements

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