

Left and Right Quotient Sets in Non-Abelian Groups

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Background

Definitions

The famous More-Sums-Than-Differences (MSTD) problem asks if $|A + A| > |A - A|$ for $A \subseteq \mathbb{Z}$, where $A + A$ denotes the set of all distinct sums and $A - A$ is the set of all distinct differences formed by the elements of A .

Definitions

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In 2006, Martin and O'Bryant proved that a positive percentage of sets are sum-dominant. [MO06]

Related problem

What if A is non-commutative?

$$AA^{-1} \neq A^{-1}A.$$

Definitions

Definition

Right Quotient Set:

- $AA^{-1} := \{a_i \cdot a_j^{-1} : a_i, a_j \in A\}$

Left Quotient Set:

- $A^{-1}A := \{a_i^{-1} \cdot a_j : a_i, a_j \in A\}$

Free groups

Definition

Let $A = \{x, y\}$ be generators. $F(A) = F_2$ is the group of **words** on $\{x, y, x^{-1}, y^{-1}\}$.

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$$x * y = xy.$$

Graph of F_2

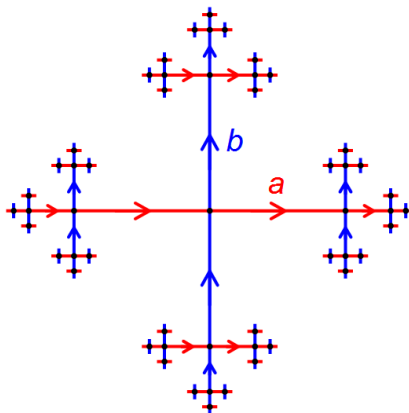


Figure: Visualization of group multiplication in F_2

Example (Quotient set example)

Example

	x	y
y^{-1}		
x^{-1}		

Table: “Multiplication table” of $A = \{x, y\}$ with $A^{-1} = \{x^{-1}, y^{-1}\}$

Definitions

Definition

A set $A \subseteq G$ is either

- **More Right Then Left (MRTL):** $|AA^{-1}| > |A^{-1}A|$.
- **Balanced:** $|AA^{-1}| = |A^{-1}A|$.
- **More Left Then Right (MLTR):** $|A^{-1}A| > |AA^{-1}|$.

Motivation

Question: What are the possible values of $|AA^{-1}| - |A^{-1}A|$ for finite subsets $A \subseteq G$?

Example

If G is abelian, then $AA^{-1} = A^{-1}A$. Thus, the only possible value of $|AA^{-1}| - |A^{-1}A|$ is 0.

History

Tao remarks

there is no relation between the size of AA^{-1} and $A^{-1}A$ in general. For instance, if H is a multiplicative set which is also a subgroup of G , and $A := (x \cdot H) \cup H$ for some x not in the normaliser of H , then AA^{-1} has about the same size as H , but $A^{-1}A$ can be much larger. [Tao11]

The Difference Graph

Motivating example

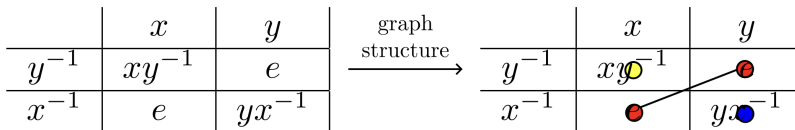


Figure: Associated to the “multiplication table” of $\{x, y\}$ is a graph.

Definition

Definition

Given a finite subset $A \subseteq G$ with $|A| = n$, the **difference graph** $D_A = (V, E)$ is defined as follows.

- Vertex set is $V := [n] \times [n]$
- Edge set is $E(D_A) := [(i, j), (k, \ell)] \iff a_i a_j^{-1} = a_k a_\ell^{-1}$

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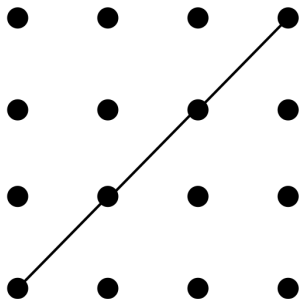
Let $\mathcal{C}(D_A)$ be the set of connected components of D_A and $c(D_A) = |\mathcal{C}(D_A)|$ be the number of connected components.

Note that $c(D_A) = |AA^{-1}|$.

Properties of D_A

	a_1	a_2	a_3	a_4
a_4^{-1}	●	●	●	●
a_3^{-1}	●	●	●	●
a_2^{-1}	●	●	●	●
a_1^{-1}	●	●	●	●

Properties of D_A



$$\longleftrightarrow a_1 a_1^{-1} = a_2 a_2^{-1} = e$$

Figure: Diagonal edges are always present.

Properties of D_A

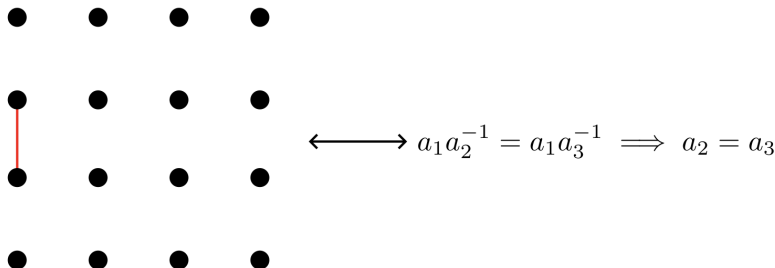


Figure: Any edge parallel to the x or y axis is not present.

Properties of D_A

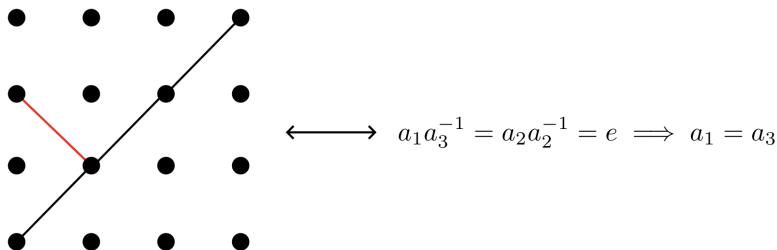
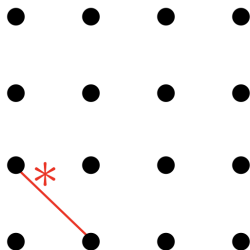


Figure: No vertex connects to the diagonal.

Properties of D_A



$$\longleftrightarrow a_1 a_2^{-1} = a_2 a_1^{-1} \implies (a_1 a_2^{-1})^2 = e$$

Figure: If G has no elements of order 2, no vertex connects to its “transpose”

Properties of D_A

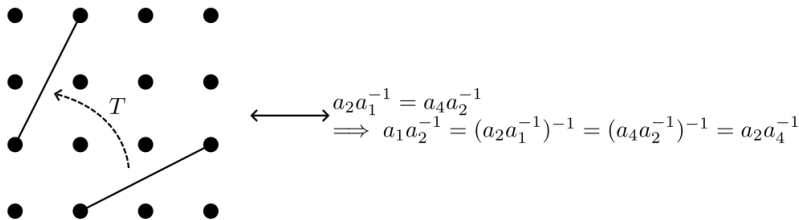
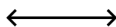
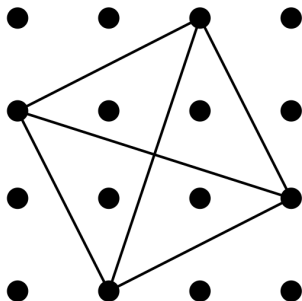


Figure: If any edge is present, its “transpose” is also present.

Properties of D_A



“=” is an
equivalence relation

Figure: Connected components form a clique.

Summary of properties

Lemma

Let $i, j, k, \ell \in [n]$.

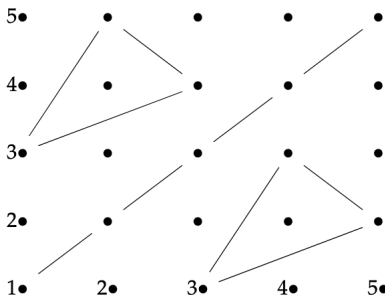
- ① The following edges are forbidden in $E(D_A)$:
 - ① $[(i, j), (k, k)]$: an edge connecting to the diagonal, provided that $i \neq j$.
 - ② $[(i, j), (i, k)]$ (or $[(j, i), (k, i)]$, but this is the handled by (1)): an edge connecting vertices on the same axis.
 - ③ **If G has no elements of order 2**, $[(i, j), (j, i)]$: an edge connecting to its symmetric pair, provided that $j \neq i$.
- ② $[(i, i), (j, j)] \in E(D_A)$.
- ③ $[(i, j), (k, \ell)] \in E(D_A) \iff [(j, i), (\ell, k)] \in E(D_A)$.
- ④ If C is a connected component in D_A , then C is a clique.

Example

For $n = 5$, $A =$

$$\{x^2, (xy)^{-1}, (xy)^{-1}x(xy)^{-1}, x(xy^{-1}), (xy)^{-1}(xy^{-1})\} \subseteq F_2$$

has the following difference graph:



Bijection of Edges

Using the fact:

$$a_i a_j^{-1} = a_k a_\ell^{-1} \iff a_k^{-1} a_i = a_\ell^{-1} a_j$$

We can obtain a **bijection of edges**

$$\begin{aligned} \phi: E(D_A) &\rightarrow E(D_{A^{-1}}) \\ [(i, j), (k, \ell)] &\mapsto [(k, i), (\ell, j)]. \end{aligned}$$

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Remark

*Tao uses this to prove the **additive energies** $\Lambda(A, A^{-1})$ and $\Lambda(A^{-1}, A)$ are equal.*

Bijection of Edges Example

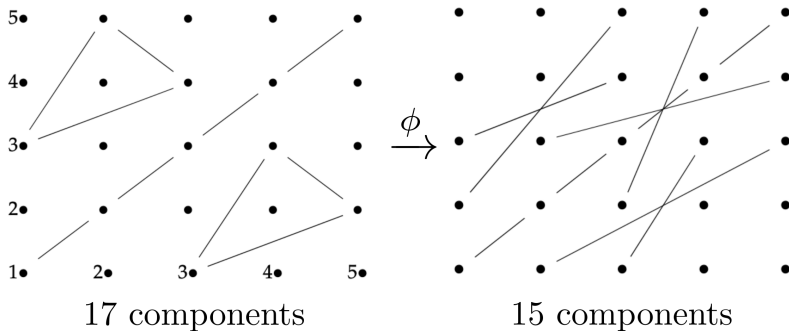


Figure: ϕ reduces the number of connected components

Nonzero Cardinality Difference

Nonzero Cardinality Difference

Theorem (SMALL 2025)

Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

- Casework: $|A| = 1, 2, 3$.

No Elements of Order 2

Definition

Recall that a group G is said to have **no elements of order 2** if for every element $g \in G$ with $g \neq e$ we have $g^2 \neq e$.

An example is $\mathbb{Z}/3\mathbb{Z}$.

No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

No Elements of Order 2

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Proof sketch.

We prove a stronger result: $|AA^{-1}|$ is always odd. A connected component is either (1) symmetric under transposition, or (2) is disjoint from its transpose. Only one component is in the first class: the diagonal. All other components come in pairs.

$$\underbrace{\text{odd}}_{(1)} + \underbrace{\text{even}}_{(2)} = \text{odd.}$$



No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq \blacksquare$.

Guesses?

Hint: Without restriction, $|A| \geq 4$ (previous theorem).

No Elements of Order 2

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 5$.

Lemma (SMALL 2025)

Let $|A| = n$. Then D_A has no connected component (other than the diagonal) with more than n elements.

- By Contradiction.

Continuation

Lemma (SMALL 2025)

Suppose $|A| = 4$. If group G does not have an element of order 2 and its largest possible cycle is K_4 , then the number of connected components in D_A is equal to the number of connected components in $D_{A^{-1}}$.

- The difference graph is heavily used to this lemma.
- We do casework on triangles.
- ϕ leaves the number of connected components unchanged.

The Free Group on 2 Generators

The Free Group on 2 Generators

Theorem (SMALL 2025)

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that
 $|A_n A_n^{-1}| - |A_n^{-1} A_n| = 2n$.

The Free Group on 2 Generators

Theorem (SMALL 2025)

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = 2n$.

The following set in F_3 with $n = 1$:

$$A := \{x, y^{-1}, y^{-1}xy^{-1}, xz, y^{-1}z\}$$

has

$$|AA^{-1}| - |A^{-1}A| = 2.$$

The Free Group on 2 Generators

Theorem (SMALL 2025)

For all $n \in \mathbb{Z}$, there exists a set $A_n \subseteq F_2$ such that $|A_n A_n^{-1}| - |A_n^{-1} A_n| = 2n$.

More generally for $n \geq 1$, A_n is constructed as a subset of $F_{3n} = F(\{x_1, y_1, z_1, \dots, x_n, y_n, z_n\})$ as follows:

$$A_n := \bigcup_{i=1}^n \{x_i, y_i^{-1}, y_i^{-1} x_i y_i^{-1}, x_i z_i, y_i^{-1} z_i\}$$

We prove

$$|A_n A_n^{-1}| - |A_n^{-1} A_n| = 2n.$$

The Infinite Dihedral Group

Definition

Definition

The **Infinite Dihedral group**, denoted D_∞ is defined as followed: $D_\infty := \langle r, s \mid s^2 = e, srs = r^{-1} \rangle$

- r : a generator representing translation
- s : a generator representing reflection

Key Properties:

- Non-abelian: $rs \neq sr$
- Has elements of order 2

The Infinite Dihedral Group

Theorem (SMALL 2025)

For every $n \in \mathbb{Z}$, there exists a subset $A_n \subseteq D_\infty$ such that
 $|A_n A_n^{-1}| - |A_n^{-1} A_n| = n$

The Infinite Dihedral Group

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Proof sketch.

D_∞ has two copies of $\mathbb{Z}$: $\langle r \rangle$ and $s\langle r \rangle$.

The Infinite Dihedral Group

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Proof sketch.

D_∞ has two copies of $\mathbb{Z}$: $\langle r \rangle$ and $s\langle r \rangle$. Let $B \subseteq \mathbb{Z}$ be finite and let $A = \{r^b, sr^b : b \in B\}$. Then

$$|AA^{-1}| - |A^{-1}A| = |B - B| - |B + B|.$$

A result of Martin and O'Bryant says this ranges over all $n \in \mathbb{Z}$. [MO06]. □

Summary Results

Results

Theorem (SMALL 2025)

Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

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Let G be a group. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 4$.

This is sharp: the quasidihedral group of order 16 has a subset of size 4 satisfying the above theorem.

Results

Theorem (SMALL 2025)

Let G be a group *with no elements of order 2*. Let $A \subseteq G$ be a finite subset. If $|AA^{-1}| \neq |A^{-1}A|$, then $|A| \geq 5$.

Results on possible values

Question: possible values of $|AA^{-1}| - |A^{-1}A|$?

Theorem (SMALL 2025)

Let G be a group with no elements of order 2. Let $A \subseteq G$ be a finite subset. Then $|AA^{-1}| - |A^{-1}A|$ is even.

Acknowledgements

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