Prime Walk to Infinity in $\mathbb{Z}[\sqrt{2}]$

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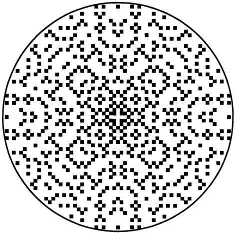

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- What if we consider $\mathbb{Z}[\sqrt{m}]$ for any integer m instead of just $\mathbb{Z}[i]$?

If $m < 0$, the ring has only finitely many units but if $m > 0$, the ring would have infinitely many units. So the number of prime elements would differ a lot.

Primes in $\mathbb{Z}[\sqrt{2}]$

Since the number of primes is much larger in $\mathbb{Z}[\sqrt{m}]$ for positive m, we conjecture it's more possible to have a prime walk in these rings. We start with the easiest one $\mathbb{Z}[\sqrt{2}]$.

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Definition

The norm of an element $a + \sqrt{2}b \in \mathbb{Z}[\sqrt{2}]$ is $|a^2 - 2b^2|$. Two elements are associates if they have the same norm. We define **standard prime** in $\mathbb{Z}[\sqrt{2}]$ as the following forms with minimal Euclidean norm:

- $\sqrt{2}$
- $a + \sqrt{2}b$ such that $a^2 - 2b^2$ is a real prime $\equiv 1, 7 \mod 8$
- $a + \sqrt{2}b$ such that $b = 0$ and a is a real prime $\equiv 3, 5 \mod 8$

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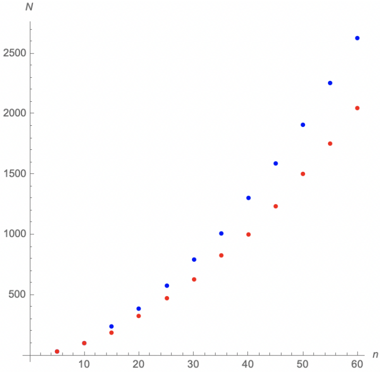
And **regular primes** or **primes** are these standard primes with associates.

Notice that primes in $\mathbb{Z}[\sqrt{2}]$ has a 4-fold symmetry.

Comparison between $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$

The figure below shows the number of primes in a disk of radius n in $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{2}]$. Blue points are in $\mathbb{Z}[\sqrt{2}]$ and red points are in $\mathbb{Z}[i]$.



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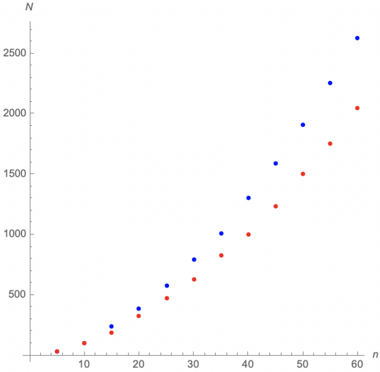
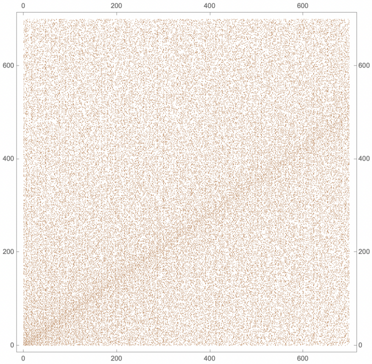


Clearly, primes grows faster in $\mathbb{Z}[\sqrt{2}]$, which motivates us to study prime walks in this ring.

Visualization of Standard Primes

Each prime of the form $a + \sqrt{2}b$ is expressed as a dot $(a, b) \in \mathbb{R}^2$. The figure below shows all the standard primes for $x, y \leq 800$. Most primes seems to cluster along the asymptote $x - \sqrt{2}y = 0$.



Conjecture

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There exists some finite step size k such that there exists an unbounded walk along the primes in $\mathbb{Z}[\sqrt{2}]$, where $\forall n\ d(p_{n+1}) > d(p_n)$, with d being the Euclidean distance.

Approaches

- Random Model
- Exhaustive Moat-finding

Random Model

Idea

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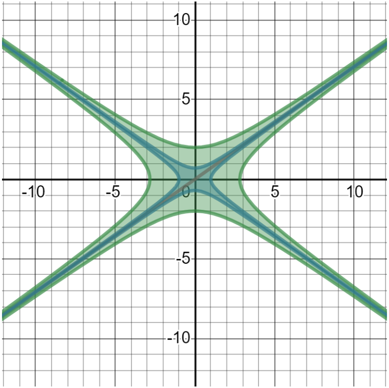
This should allow us to then look at the long term behavior of walks when it becomes too computationally expensive to generate all walks.

Estimating the Number of Primes - Preliminaries

- We consider primes in the region $|a^2 - 2b^2| \leq r^2$, which straddles the asymptotes. This is a generalization of the disk region for Gaussian primes, using the norm of $\mathbb{Z}[\sqrt{2}]$.
- This region is unbounded as it approaches the asymptotes.



Estimating the Number of Primes - Extending PNT

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Prime Number Theorem in $\mathbb{Z}[\sqrt{2}]$

The number of primes in $\mathbb{Z}[\sqrt{2}]$ within the region $|a^2 - 2b^2| \leq r^2$ is about $\frac{4r^2}{\log r} + \frac{r}{\log r} + 2$.

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Prime Number Theorem in $\mathbb{Z}[i]$

The number of Gaussian primes in the disk of radius r about the origin is $\frac{2r^2}{\log r} + \frac{2r}{\log r} + 4$.

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- If C is centered d from the origin, our estimate is $\left(\frac{(d+r)^2}{\log(d+r)} - \frac{(d-r)^2}{\log(d-r)}\right) \frac{r}{2d}$

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- As before, any solution z belongs to a family of solutions $z(1+\sqrt{2})^{2n}$ which maintain the norm.

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Theorem (Bernays)

Let $f(x, y) = rx^2 + sxy + ty^2$ be defined on the integers with integer coefficients such that $s^2 - rt$ is not square. Then the number of positive integers less than n that can be expressed as $f(x, y)$ is $O\left(\frac{n}{\sqrt{\log n}}\right)$

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Theorem (Polymath 2020)

$\exists\ a_1, b_1 \in \mathbb{Z}$ such that $a_1^2 - 2b_1^2 = c \iff \exists\ a_2, b_2 \in \mathbb{Z}$ such that $a_2^2 - 2b_2^2 = -c$

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For comparison,

- In $\mathbb{Z}$: $O\left(\frac{1}{\log n}\right)$
- In $\mathbb{Z}[i]$: $O\left(\frac{1}{\log n}\right)$

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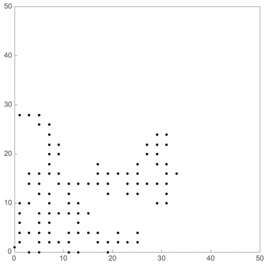

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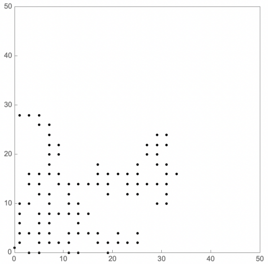

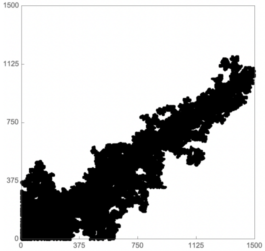

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Using this lemma, we are able to prove the next theorem.

Number of Norms

Theorem (Polymath 2020)

It's impossible to walk to infinity using primes of only finitely many norms, or integer solutions to only finitely many generalized Pell's equations.

Number of Norms - Proof Sketch

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- Then the number of primes between this gap should be at least $O(b)$.
- By Lemma 1, we must have solutions to infinitely many Pell's equation to finish the gap as $b \to \infty$.

Boundedness from the Asymptote

Theorem (Polymath 2020)

It is impossible to perform a walk of bounded step size to infinity in $\mathbb{Z}[\sqrt{2}]$ if we remain within some bounded distance from the asymptote $y = \frac{1}{\sqrt{2}}x$.

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- The number of norm-curves with norm $\pm 1 \pmod 8$ we could possibly access at x is $O\left(\frac{x}{\log x}\right)$.

- We expect one prime for each such curve since associates grow exponentially, thus the number of steps grows faster than the number of options for steps!

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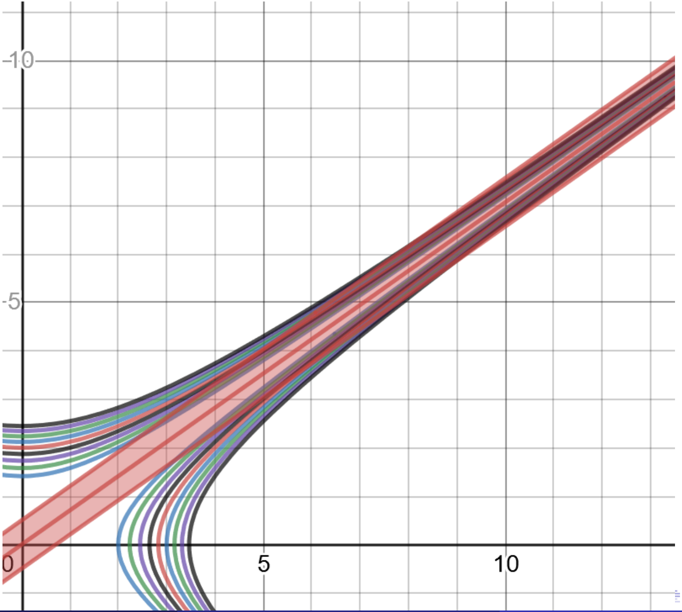

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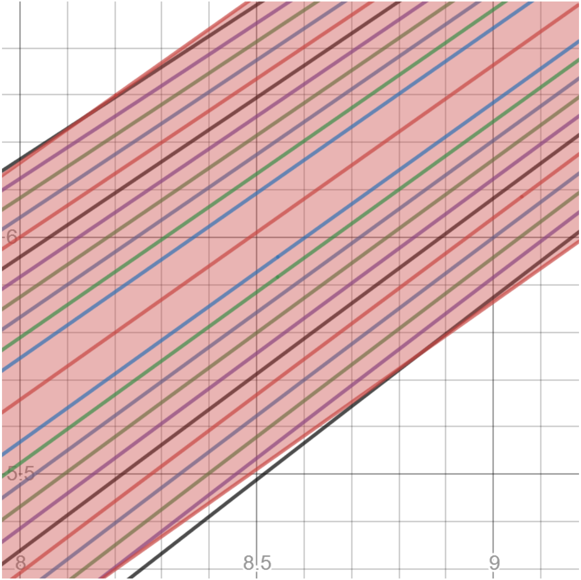

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