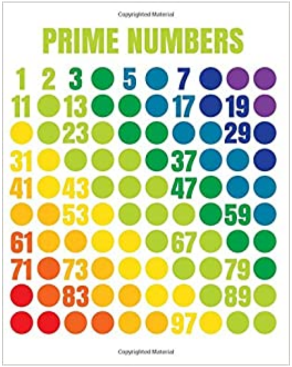
Walking to Infinity Along Some Number Theory Sequences

Tudor Popescu, Saam Rasool, Ploy Wattanawanichkul
(joint work with Steven J. Miller, Fei Peng, Joshua M. Siktar, and Polymath REU Prime Walk group)

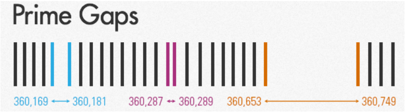
September 19, 2020

Outline

1. Introduction
2. Disproved conjectures
3. Behavioral models
   - Prime walks
   - Square-free walks
4. Proofs
   - Prime Walks
   - Fibonacci Walks
5. Conclusion



Introduction



- **Prime gaps**: How large could it be?
  $(n+1)!+2, (n+1)!+3, (n+1)!+4, \ldots, (n+1)!+n+1$.
- **A more interesting open problem:** Is it possible to walk to infinity by appending a bounded number of digits to a prime at each stage while staying prime? $2, 23, 233, \ldots$
- What about **square-free** numbers? or **Fibonacci** numbers?
- Prime/Square-free numbers $\Longrightarrow$ stochastic models.
- Fibonacci $\Longrightarrow$ No
- Primes in base $2, 4, 5$? No

Disproved conjectures

Definition

$\mathrm{SF} := \{x \in \mathbb{Z}^+ : \forall k > 1 \in \mathbb{Z}^+, k^2 \nmid x\}$.

Definition

$\mathrm{NSFE}_b := \{x \in \mathbb{Z}^+ : \overline{xi} \cap \mathrm{SF} = \emptyset, \forall i \in \{0, \ldots, b-1\}\}$. *Such numbers are not square-free extendable in base b.*

Definition

$\mathrm{RTSF}_b := \{\overline{xi} \in \mathrm{SF} | x \in \mathrm{SF}\}$. *Such numbers are right truncatable square-free, meaning that they remain square-free when the last digit is successively removed.*

Disproved conjectures

Conjecture (disproved)

$\mathrm{SF} \cap \mathrm{NSFE}_{10} = \emptyset; \mathrm{RTSF}_{10} \cap \mathrm{NSFE}_{10} = \emptyset.$

- These conjectures turn out to be false, however, as can be seen in the following examples.

Example

$23154621017069422 \in \mathrm{RTSF}_{10} \cap \mathrm{NSFE}_{10}$.

- For left-appending, 91169368838469843635793 is square-free, but $\overline{i91169368838469843635793}$ never is.

Remark

That means, it is possible to start with the empty string, append one digit at a time, and reach an end.

Outline

Behavioral models of walks; Prime walking to infinity

- The prime number theorem states that the number of primes that are less than or equal to n is asymptotically $\pi(n) \approx \frac{n}{\log(n)}$. Our model assumes that the probability that n is prime is $\frac{1}{\log(n)}$. We approximate that the probability that a $k-$digit number is prime is $\frac{1}{k\log(10)}$.
- Some simple probability manipulations yields that

$$\sum_{n=0}^{\infty}\prod_{k=1}^{n-1}\left(1-\left(1-\frac{1}{k\log(b)}\right)^{b}\right)=\sum_{n=1}^{\infty}\mathbb{P}[\text{walk has length at least n}]=$$

$$=\sum_{n=1}^{\infty}n\mathbb{P}[\text{walk has length exactly n}]=\mathbb{E}[X]$$

Behavioral models of walks; Prime walking to infinity

- Multiplying by the approximate number of primes with exactly r digits and dividing by the expected number of primes with *at most* s digits, we get that the expected length of a walk with starting point at most s digits is about

$$\frac{s(b-1)}{b^s}\left(\sum_{r=1}^{s}\frac{b^{r-1}}{r}\left(\sum_{n=0}^{\infty}\prod_{k=r}^{n-1}\left(1-\left(1-\frac{1}{k\log(b)}\right)^{b}\right)\right)\right)$$

Base	Number of digits of starting point 1	2	3	4	5	6	7
2	5.20	9.90	11.62	11.45	10.40	9.08	7.79
3	5.05	7.75	7.60	6.53	5.40	4.49	3.80
4	4.87	6.55	5.86	4.79	3.92	3.29	2.85
5	4.71	5.79	4.92	3.96	3.25	2.78	2.45
6	4.57	5.27	4.34	3.48	2.89	2.49	2.22
7	4.46	4.89	3.95	3.17	2.65	2.31	2.08
8	4.37	4.59	3.67	2.95	2.49	2.19	1.98
9	4.29	4.36	3.45	2.79	2.37	2.09	1.91
10	4.22	4.17	3.28	2.66	2.28	2.20	1.85
10'	4.54	4.55	3.55	2.83	2.38	2.09	1.90

Table: Expected length of prime walks given by our formula.

Behavioral models of walks; Prime walking to infinity

Start has x digits	1	2	3	4	5	6
greedy model	1.89	1.60	1.41	1.30	1.25	1.20
refined greedy model	4.33	3.37	2.76	2.37	2.08	1.90
primes	8.00	4.71	3.48	2.71	2.28	2.03

Table: Comparing the expected value of the walk lengths. The refined greedy model is significantly closer to the actual value compared to the greedy one.

Number of appended	1's	3's	7's	9's
random model	15.4%	32.7%	18.5%	33.2%
refined greedy model	12.5%	35.9%	14.7%	36.8%
primes	13.1%	38.8%	12.2%	35.6%

Table: Frequency of added digits in prime walks with starting point less than 1,000,000.

Start has x digits	1	2	3	4	5	6
greedy model	2.83	1.94	1.64	1.45	1.34	1.28
refined greedy model	3.49	3.22	2.43	2.04	1.77	1.62
primes	8.00	3.81	2.64	2.12	1.81	1.64

Table: Expected value of the walks with starting point 2 modulo 3.

Behavioral models of walks; Prime walking to infinity

- If we are allowed to add a digit anywhere, The expected value for walk length for an "m" digit long number is:

$$\sum_{n=1}^{\infty} \prod_{k=m}^{n-1} \left(1 - \left(1 - \frac{1}{k \log(b)}\right)^{b(k+1)-1-k}\right)$$

Table: Expected value for small starting lengths evaluated up to $n = 1000$

	starting length									
	1	2	3	4	5	6	7	8	9	10
2	6.22	6.74	5.89	5.35	4.99	4.73	4.54	4.40	4.28	4.18
3	10.01	9.01	8.25	7.74	7.37	7.09	6.88	6.71	6.58	6.46
4	13.32	12.33	11.55	10.99	10.58	10.26	10.01	9.80	9.63	9.49
5	17.56	16.57	15.76	15.16	14.69	14.33	14.03	13.79	13.58	13.40
6	22.90	21.90	21.07	20.42	19.90	19.49	19.15	18.87	18.63	18.42
7	29.59	28.59	27.73	27.04	26.48	26.03	25.65	25.32	25.05	24.81
8	37.96	36.97	36.08	35.36	34.76	34.26	33.85	33.49	33.17	32.90
9	48.45	47.45	46.55	45.79	45.16	44.63	44.17	43.78	43.43	43.13
10	61.57	60.57	59.65	58.87	58.21	57.64	57.15	56.72	56.35	56.01

- At $b = 10, k = 1$, we have an expected walk length of 61.57, large compared with the expected value for appending on the right only.

Behavioral models of walks; Prime walking to infinity

- Some example that we found is of length 17: {7, 17, 137, 1637, 18637, 198637, 1986037, 19986037, 199860337, 1998660337, 19998660337, 199098660337, 1949098660337, 19490986560337, 194909865603317, 1949098656033817, 19490983656033817.}
- The expected walk length is not bounded as base increases. It seems highly likely that a walk to infinity is possible by adding a digit anywhere.
- Using the Miller-Rabin primality test we see that walks of over length 100 in base 10 are extremely common. We can find examples of such walks using this method; even though they are not exhaustively checked, we expect them to be correct with margin of error 4^{-40}.

Outline

Behavioral models of walks; Square-free walking to infinity

- A square-free integer is an integer that is not divisible by any perfect square other than 1. If $Q(x)$ denotes the number of square-free positive integers less than or equal to x, then we have that

$$Q(x) \approx x \prod_{\text{p prime}} (1 - \frac{1}{p^2}) = x \prod_{\text{p prime}} \frac{1}{1 + \frac{1}{p^2} + \frac{1}{p^4} + \ldots} = \frac{x}{\zeta(2)} = \frac{6x}{\pi^2}.$$

- Using this, our model assumes that the probability that x is square-free is $p = \frac{6}{\pi^2}$.
- Let $\mathbb{X}$ denote the number of steps in our random square-free walk.
- Since $\mathbb{X}$ is a geometric random variable, it is easy to see that

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} kp^k(1-p) = \frac{p}{1-p} = \frac{6}{\pi^2 - 6} \approx 1.55.$$

Numerical results

- As previously stated, the expected number of steps our model takes is 1.55
- However, when starting with small values, intuitively we should have longer walks as the primes are sparse. Computer simulations suggest this as well
- When we start with a number between 1 and 100, the expected value tends to 1.77;
- When we start with a number between 1000 and 100000, the expected value tends to 1.71;
- When we start with a number between 1000000 and 100000000, the expected value tends to 1.69;
- when we start with a number between 10000000000 and 1000000000000, the expected value tends to 1.52;

Numerical results

Digit added	Number of digits of starting point 1	2	3	4	5	6
0	10.1%	7.4%	7.6%	7.5%	7.5%	7.5%
1	14.0%	13.6%	13.2%	13.4%	13.4%	13.4 %
2	8.4%	5.5%	5.3%	5.3%	5.3%	5.3%
3	13.5%	13.5%	13.4%	13.4%	13.4%	13.3%
4	5.1%	8.1%	8.0%	8.0%	8.0%	8.0%
5	12.1%	10.8%	10.9%	10.8%	10.8%	10.8%
6	8.3%	5.5%	5.4%	5.3%	5.3%	5.3%
7	13.4%	13.5%	13.2%	13.3%	13.3%	13.3%
8	4.9%	7.4%	8.0%	8.0%	8.0%	8.0%
9	9.7%	14.2%	14.5%	14.6%	14.6%	14.6%

Table: Comparing the frequency of the digits of square-free walks in base 10.

Remarks

- Odd digits appear more often than even digits; this is because if x is square-free, then it cannot be a multiple of 4, hence even digits appear less.
- The frequencies of 2 and 6 are less than the frequencies of $0, 4$, and 8. This is because if x and $\overline{xi}$ are square-free and i is even, then if x is odd, by modulo 4 considerations i is $0, 4$, or 8, and if x is even, then i is 2 or 6. However, x is almost twice more likely to be odd, hence the frequency of $0, 4, 8$ is bigger than that of 2 and 6.
- 5 appears less than any other odd digit; similar to the above, $\overline{x5}$ isn't square-free if x ends with 2 or 7.
- 9 appears more often than any other digit; this is because if x is square-free, then $x \not\equiv 0 \pmod 9$, hence $\overline{x9} \not\equiv 0 \pmod 9$;
- As the starting point increases, the frequencies stabilize.

Outline

Appending In Different Bases

Theorem

It is impossible to walk to infinity on primes in base $2, 4$ and 5 by appending one digit at a time to the right.

Proof (Base 4).

We only consider append 1 or 3. Since appending 1 to a prime $p \equiv 2 \pmod 3$ gives $4p + 1 \equiv 0 \pmod 3$, one can append 1 at most a single time in walking to infinity. Thus, it suffices to consider the infinite subsequence over which only 3's are appended.

$$\begin{aligned} p_2 &= 4p_1 + 3, \\ p_3 &= 16p_1 + 15, \ldots \\ p_i &= 4^{i-1}p_1 + 4^{i-1} - 1. \end{aligned}$$

However, $p_{p_1} \equiv 4^{p_1-1}p_1 + 4^{p_1-1} - 1 \equiv 0 \pmod{p_1}$, by Fermat's little theorem. Hence it is impossible to walk to infinity. □

Prime Walks: Appending An Unbounded Number of Digits

Theorem (Left Appending)

It is possible to construct an infinite prime walk by appending unbounded number of digits to the left.

- 3, 83, 1483, 11483, ...
- Dirichlet's theorem on arithmetic progressions: given $(a, d) = 1$, there are infinitely many $a \pmod{d}$ primes, i.e., in the sequence $a, a + d, a + 2d, a + 3d, \ldots$
- Thus given an initial prime p_0 other than 2 or 5, we can take n such that $10^n > p_0$, and find a prime p_1 such that $p_1 \equiv p_0 \pmod{10^n}$.

Theorem (Right Appending)

It is possible to construct an infinite prime walk by appending unbounded number of digits to the right.

Prime Walks: Appending An Unbounded Number of Digits

Proof (Right Appending).

Given $p \in \mathbb{P}$, we must show that there exists an n such that there is a prime between $10^n \cdot p$ and $10^n \cdot p + 10^n - 1 = 10^n(p+1) - 1$. Note that for a given p, and for a fixed $r \in \mathbb{R}$, $0 < r < 1$, there exists an n such that

$$p < 10^{\frac{1-r}{r}n} - 1 \implies p10^n < 10^{\frac{n}{r}} - 10^n.$$

Moreover, given such an n, then it is possible to find $x \in \mathbb{R}^+$ such that

$$p10^n = x - x^r \implies x - x^r < 10^{\frac{n}{r}} - 10^n.$$

Thus, $x^r < 10^n$, since $x - x^r$ is strictly increasing for positive values. Given that $x^r < 10^n$, then $x - x^r > x - 10^n \implies p10^n > x - 10^n$, and so $x < (p+1)10^n$. Thus, given fixed p, r, we could find n such that there exists $x > c$ and $[x - x^r, x] \subset [p10^n, (p+1)10^n)$ for some constant c. According to [BHP], there exists a prime in $[x - x^{0.525}, x]$ for any $x > x_0$. Note that, to guarantee $x > x_0$, choose $n > \log_{10}((x_0 - x_0^r)/p)$. □

Outline

Fibonacci Walks

Lemma

- $\forall m, k \in \mathbb{N},\ F_{k+1}F_m \leq F_{m+k} \leq F_{k+2}F_m.$
- $\forall m > k \in \mathbb{N}, k > 2,\ F_{m+k} = (F_{k+2} - F_{k-2})F_m + (-1)^{k+1}F_{m-k}.$

Theorem (Appending Exactly One Digit)

It is impossible to have an infinite Fibonacci walk by appending one digit at a time. In particular, all such walks have length at most 2.

Theorem (Appending Exactly N Digits)

It is impossible to have an infinite Fibonacci walk by appending $N \in \mathbb{N}$ digits at a time. In particular, any appendable step must be $\leq \frac{8 \cdot 10^N - 8}{7}$.

Any appendable step is at most $\log \frac{8 \cdot 10^N - 8}{7} \approx N$ digits. Thus, we can append N digit only once, so the walk must have length at most 2.

Fibonacci Walks

Theorem (Appending At Most N digits)

Given we can append at most N digits each time and the starting number contains N_0 digits, the length of the longest walk is at most $\lfloor \log_2 \frac{N}{N_0} \rfloor + 2$.

Proof.

Starting with F_0 with N_0 digits, $10^{N_0-1} \leq F_0 \leq 10^{N_0} - 1$. From Appending Exactly N Digits theorem, $10^{N_0-1} \leq F_0$ tells us that we cannot append $0, 1, \ldots, N_0 - 1$ digits to F_0. Thus, we can only append N_0 digits or above to F_0. By the same analysis, we are required to add at least $2^{M-1}N_0$ digits at the M-th step. Hence, we can determine the largest M.

$$2^{M-1}N_0 \leq N \Longrightarrow M \leq \log_2(N/N_0) + 1$$

Therefore, the length of the longest walk is at most $\lfloor \log_2 \frac{N}{N_0} \rfloor + 2$ including the last number that cannot be appended. □

Conclusion

1. Prime

Appending an unbounded number of digits, both to the left and right	proven possible
Appending one digit at a time to the right	believe impossible
Appending one digit at a time anywhere	believe possible

2. Square-free

Appending one digit at a time to the right	believe possible

3. Fibonacci

Appending a bounded number of digits to the right	proven impossible

Bibliography

Normal Number. (n.d.). Retrieved from http://mathworld.wolfram.com/NormalNumber.html

Trucnatable primes. (n.d.). Retrieved from https://primes.utm.edu/glossary/page.php?sort=RightTruncatablePrime

Baker, R. C., Harman, G., & Pintz, J. (2001). The difference between consecutive primes, II. Proceedings of the London Mathematical Society, 83(3), 532-562.