Low Lying Zeros of Number Field L-functions

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Young Mathematicians Conference
The Ohio State University, August 29, 2009

Random Matrices and Number Theory

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1\le i\le j\le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1\le i\le j\le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij})dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

Random Matrix Theory: Eigenvalue Trace Formula

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\sum_{n=1}^{N} \lambda_i(A)^k \ = \ \mathrm{Trace}(A^k),$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

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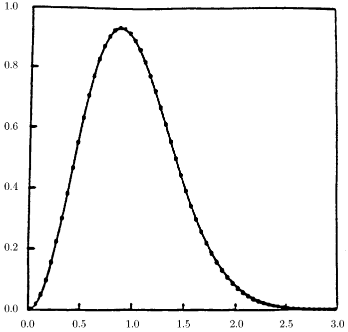
Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

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Measures of Spacings: 1-Level Density

$\phi(x)$ even Schwartz function whose Fourier Transform is compactly supported.

1-level density

$$D_f(\phi) = \sum_j \phi\left(L_f \gamma_{j;f}\right)$$

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.

ϕ decays too rapidly for this sum to be evaluated asymptotically.

Families of L-functions

Instead of looking at a single L-function $L(s, f)$, need to average over a family $\{L(s, f) : f \in \mathcal{F}\}$ of 'similar' L-functions.

1-level density for families

$$\begin{aligned} D_{\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_f(\phi) \\ &= \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{j} \phi\left(L_f \gamma_{j;f}\right) \end{aligned}$$

Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the 1-level density depends only on a symmetry group attached to the family.

Number Fields

Definitions

A *number field* is a subfield K of $\mathbb{C}$ which forms a finite-dimensional vector space over $\mathbb{Q}$.

Examples

1. $\mathbb{Q}(\sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Q}\}$
2. $\mathbb{R}, \mathbb{C}, \bar{\mathbb{Q}}$ are not number fields

Definitions

Every number field $K/\mathbb{Q}$ contains a *ring of algebraic integers* $\mathcal{O}_K$ with properties similar to those of $\mathbb{Z}$.

Examples

1. $K = \mathbb{Q}(\sqrt{-5})$. Have $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, and
$$6 = 2 \cdot 3 = (1 - \sqrt{-5})(1 + \sqrt{-5})$$
2. $K = \mathbb{Q}(i)$. Have $\mathcal{O}_K = \mathbb{Z}[i]$, and
$$2 = (1 - i)(1 + i)$$
3. $K = \mathbb{Q}(\sqrt{-3})$. Have $\mathcal{O}_K = \mathbb{Z}\left[\frac{1 + \sqrt{-3}}{2}\right]$.

Prime ideals

Because of non-unique factorization, doesn't help to study prime elements of $\mathcal{O}_K$.

Theorem (Unique factorization of ideals)

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$. Then $\mathfrak{a}$ factors uniquely as a product of prime ideals:

$$\mathfrak{a} = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r}.$$

Thus, we study prime ideals instead of prime elements.

Prime ideals

Fact: Every non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is maximal. Thus $\mathcal{O}_K/\mathfrak{p}$ is a field.

Lemma

The field $\mathcal{O}_K/\mathfrak{p}$ is finite.

We define $N\mathfrak{p} := |\mathcal{O}_K/\mathfrak{p}|$. If $\mathfrak{a}$ is an arbitrary ideal of $\mathcal{O}_K$, define

$$N\mathfrak{a} = N\mathfrak{p}_1^{e_1} \cdots N\mathfrak{p}_r^{e_r}.$$

Dedekind zeta function

Given a number field K, we define for Re(s)>1

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s}.$$

Have an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1}.$$

How to attach L-functions to K?

Number Field L-functions

Ideal class characters

An *ideal class character* is a function χ which assigns complex numbers to ideals of $\mathcal{O}_K$ in a multiplicative fashion, i.e. $\chi(\mathfrak{a}\mathfrak{b}) = \chi(\mathfrak{a})\chi(\mathfrak{b})$, and which is trivial on principal ideals.

Given such a character, define the L-function

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}.$$

How to study the zeros of this function?

Explicit formula

Need an *explicit formula*, which relates sums over zeros to sums over primes.

Weil's explicit formula

$$\sum_{L(1/2+i\gamma_\chi,\chi)=0} \widehat{\phi}(\gamma_\chi) = 4\delta_\chi \int_0^\infty \phi(x)\cosh(x/2)dx$$
$$+\phi(0)(\log\Delta - N\gamma - N\log 8\pi - \frac{r_1\pi}{2})$$
$$-\sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{m=1}^{\infty} \frac{\phi(m\log N\mathfrak{p})}{N\mathfrak{p}^{m/2}}(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m})$$
$$+r_1\int_0^\infty \frac{\phi(0)-\phi(x)}{2\cosh(x/2)}dx + N\int_0^\infty \frac{\phi(0)-\phi(x)}{2\sinh(x/2)}dx$$

Explicit formula

Gives a nice expression for the 1-level density:

$$\begin{aligned} D_\chi(\phi) &= \sum_{L(1/2+i\gamma_\chi)=0} \phi\left(\frac{\log\Delta}{2\pi}\gamma_\chi\right) \\ &= \frac{1}{\log\Delta}\left[4\delta_\chi \int_0^\infty \widehat{\phi}\left(x\frac{2\pi}{\log\Delta}\right)\cosh(x/2)dx\right. \\ &+ \widehat{\phi}(0)(\log\Delta - N\gamma - N\log 8\pi - \frac{r_1\pi}{2}) \\ &- \sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{m=1}^\infty \frac{\widehat{\phi}\left(m\frac{2\pi}{\log\Delta}\log N\mathfrak{p}\right)}{N\mathfrak{p}^{m/2}}(\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}) \\ &+ \left. r_1\int_0^\infty \frac{\widehat{\phi}(0)-\widehat{\phi}(x)}{2\cosh(x/2)}dx + N\int_0^\infty \frac{\widehat{\phi}(0)-\widehat{\phi}(x)}{2\sinh(x/2)}dx\right]. \end{aligned}$$

Class number

The 1-level density for the family is then

$$D_{\mathcal{F}}(\phi) = \frac{1}{h} \sum_{\chi} D_{\chi}(\phi)$$

where h is the *class number* of K.

Need to understand what happens to h as K changes.

Main Difficulty: If there are many prime ideals with small norm, then the sum

$$\sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{m=1}^{\infty} \frac{\widehat{\phi}\left(m2\pi\frac{\log N\mathfrak{p}}{\log \Delta}\right)}{N\mathfrak{p}^{m/2}} (\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m})$$

might blow up.

Use algebraic number theory to control norms of primes.

Results

Fouvry and Iwaniec

Fouvry and Iwaniec studied the family $\{\mathbb{Q}(\sqrt{-D})\}$ for suitable D.

$$p = \frac{m^2 + Dn^2}{4}$$
$$\Rightarrow p \geq \frac{D}{4}$$

Theorem (Fouvry and Iwaniec, 2003)

Let ϕ be an even Schwartz function whose Fourier transform is supported in $(-1, 1)$. Then for the family $\{\mathbb{Q}(\sqrt{-D})\}$, the 1-level density is

$$D_{\mathcal{F}}(\phi) \sim \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int \phi(x) W_{\mathrm{Sp}}(x) dx$$

Main result

Theorem

Let K_0 be a normal, totally real field of class number one. Then for the family $\{K\}$ of all CM-fields such that $K^+ = K_0$, the 1-level density is again given by the symplectic distribution:

$$D_{\mathcal{F}}(\phi) \sim \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int \phi(x) W_{\mathrm{Sp}}(x)dx$$

THANK YOU!