Solutions to a Pair of Diophantine Equations

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Polymath Jr. Conference 2025
August 9, 2025

We gratefully acknowledge support from the National Science Foundation DMS2341670

A pair of equations

For relatively prime $a, b \in \mathbb{N}$, consider

$$\begin{aligned} ax + by &= \frac{(a-1)(b-1)}{2} \\ 1 + ax + by &= \frac{(a-1)(b-1)}{2} \end{aligned}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a nonnegative integral solution (x, y). The solution is unique.

Fibonacci numbers

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$$(x, y) = ?$$

Chu's six cases (2020)

$$F_{6k} \cdot \boxed{\frac{F_{6k-1}-1}{2}} + F_{6k+1} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}$$

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$$1 + F_{6k+5} \cdot \boxed{\frac{F_{6k+4}-1}{2}} + F_{6k+6} \cdot \boxed{\frac{F_{6k+4}-1}{2}} = \frac{(F_{6k+5}-1)(F_{6k+6}-1)}{2}$$

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Generalize the problem to

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Cassini's identity (2): $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$

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Fibonacci pairs (3): $(F_{6n}, F_{6n+3}), (F_{6n+1}, F_{6n+4}), (F_{6n+2}, F_{6n+5})$

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Define $(t_n^{(u,v)})_{n=1}^{\infty} : t_1^{(u,v)} = u, \quad t_2^{(u,v)} = v, \quad t_n^{(u,v)} = t_{n-1}^{(u,v)} + t_{n-2}^{(u,v)}$

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Nonnegative integral $(x, y) = ?$ Depend on n modulo 6.

Sample case: $n \equiv 4 \mod 6$

Polymath Jr. 25 Diophantine Group

Given $(u, v, n, r) \in \mathbb{Z}^4$ with even n, it holds that

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2},$$

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Need r such that the solutions are nonnegative integers

Choose r when $n \equiv 4 \mod 6$

Lemma

Given $(u, v) \in \mathbb{N}^2$ with $\gcd(u, v) = 1$ and odd u,

$$\exists! \text{ odd } r \in [1, u] \text{ with } vr \equiv \pm 1 \mod u.$$

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If u is odd, $u > 1$, and $vr \equiv -1 \mod u$ or u is even and $vr \equiv -1 \mod 2u$,

$$\frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v-1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr+1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}.$$

Integral solutions

Let $r = \mathbb{O}(u, v)$ (which is odd).

If u is odd and $vr \equiv 1 \mod u$,

$$1 + \frac{1}{2}\left((u - r) \underbrace{F_{n-1}}_{even} + \underbrace{\frac{(u-r)v+1}{u}}_{odd} \underbrace{F_n}_{odd} - 1 \right) t_n^{(u,v)} +$$

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If u is even and $vr \equiv 1 \mod 2u$,

$$\frac{1}{2}\left(\underbrace{r}_{odd}\underbrace{F_{n-2}}_{odd} + \underbrace{\frac{vr-1}{u}}_{even} F_{n-1} - 1 \right)$$

Example: $u = 10, v = 3, n = 10$

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$$r = \mathbb{O}(u, v) = 7$$

$$1 + \underbrace{\frac{1}{2}\left((u-r)F_9 + \frac{(u-r)v+1}{u}F_{10} - 1\right)}_{78} 312 +$$

$$\underbrace{\frac{1}{2}\left(rF_8 + \frac{vr-1}{u}F_9 - 1\right)}_{107} 505 = \frac{(312-1)(505-1)}{2}.$$

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This matches Chu's (2020) ✓ :

$$1 + F_{6k+4} \cdot \boxed{\frac{F_{6k+4}-1}{2}} + F_{6k+5} \cdot \boxed{\frac{F_{6k+2}-1}{2}} = \frac{(F_{6k+4}-1)(F_{6k+5}-1)}{2}$$

Future investigations

Study the solutions when $(a, b) = (F_n^i, F_{n+1}^j)$ for $i, j \in \mathbb{N}$

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Study other recurrences