Zeckendorf Games

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The Zeckendorf Game

This game is introduced in "The Zeckendorf Game" paper[1]
Rules: At the beginning of the game, there is an unordered list of n 1's.
Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{F_1^n\}$.
On each turn, a player can do one of the following moves:

1. $F_{i-1} \wedge F_i \rightarrow F_{i+1}$
2. If the list has two of the same Fibonacci number, $F_i \wedge F_i$ then
   a. if $i = 1$, $F_1 \wedge F_1 \rightarrow F_2$
   b. if $i = 2$, $F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$
   c. if $i \geq 3$, $F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1}$

The game terminates at the Zeckendorf decomposition(no more moves left).

Games end

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

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Winning strategies
Previous Results

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

For all $n > 2$, Player 2 has the winning strategy for 2 player Zeckendorf Game.

- **Idea:** If not, P2 could steal P1's Winning strategy.

Result 1:

For all $n \geq 5$, $p \geq 3$ Multi-player Game, no player has winning strategy

- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy
  1. Since for all $n \geq 5$, $p \geq 3$ games, any player m's winning path does not contain the following 3 consecutive steps(unless player m is the player who takes step 2). If it contains, player in step 2 can do $F_1 \wedge F_2 \rightarrow F3$ instead and player $m-1$ can steal the winning strategy:
     Step $1: F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Step $2: F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Step $3: F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ (Split two 2s into one 1 and one 3)
  2. Then we construct other $m-1$ players' moves containing these 3 consecutive steps, which contradicts, so player m has no winning strategy

Result 2:

In a game consisting of t teams and exactly k consecutive players each team. When n is significantly large, for any $t \geq 3, k = t - 1$, no team has winning strategy

- **Idea:** Suppose team m has the winning strategy $(1 \leq m \leq t)$. Then team m-1 can steal team m's winning strategy
  1. Since for any $t \geq 3, k = t - 1$, any team m's winning path doesn't contain the following $3k$ consecutive steps (unless one of the middle k players is in team m). If it contains, the middle k players listed below can all do $F_1 \wedge F_2 \rightarrow F3$ instead and team $m - 1$ can steal the winning strategy:
     First k steps all do : $F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Middle k steps all do : $F_1 \wedge F_1 \rightarrow F_2$ (Combine two 1s into one 2)
     Last k steps all do : $F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ (Split two 2s into 1 and 3)
  2. Then we construct these $3k$ steps for other $m - 1$ teams and we get contradiction

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Bounds on game length

Previous Result

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

Lower bound on length of game: $n - Z(n)$
Upper bound on length of game: $\log_\phi(\sqrt{5}n + 1/2)n$

Theorem (Li, R., Li, X., Miller, S. J., Mizgerd, C., Sun, C., Xia, D., & Zhou, Z. (2020).*"Deterministic Zeckendorf Games"*.[2])

Upper bound on length of game: $3n - 3Z(n) - IZ(n) + 1$

Notations

$Z(n)$: number of terms in Zeckendorf Decomposition. $Z(n) = \Theta(\log n)$
$IZ(n)$: sum of indices in Zeckendorf Decomposition. $IZ(n) = \Theta(\log^2 n)$

Bounds on game length

Previous Result

Theorem (Li, R., Li, X., Miller, S. J., Mizgerd, C., Sun, C., Xia, D., & Zhou, Z. (2020).*"Deterministic Zeckendorf Games"*.[2])

The upper bound of the game is given by the sum of the three parts:

a. $MC_3 + MC_4 + \cdots + MC_{i_{\max}(n)} + MS_3 + MS_4 + \cdots + MS_{i_{\max}(n)} \leq n - IZ(n)$
b. $MC_1 + MC_2 \leq n - Z(n)$
c. $MS_2 \leq n - 2Z(n) + 1$

Notations

MC_i: number of Combine moves at F_i
i.e. $(F_1 \wedge F_1 \rightarrow F_2$ or $F_{i-1} \wedge F_i \rightarrow F_{i+1})$
MS_i: number of Split moves at F_i
i.e. $(F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ or $F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1})$
$i_{\max}$: the largest index m such that $F_m \leq n$

Bounds on game length

New Result

Result:

New bound is $\frac{\sqrt{5}+3}{2}n - \frac{\sqrt{5}+1}{2}Z(n) - IZ(n)$

- **Idea:** tight the bound of MS_2
  1. Base on the fact that there is at most one F_2 at the end of the game, find relation between MS_2 and other MC_i's and MS_i's

     $$\text{Ex: } MS_2 \leq (MC_1 - MC_2 - MC_3 + MS_4)/2$$

  2. Construct series of inequalities by replacing any $MS_i (i \geq 3)$ terms on the right hand side with similar inequalities
  3. Find patterns in the coefficients of MC_i's and MS_i's on the right hand side and evaluate the inequality for MS_2
  4. Combine the new bound on MS_2 with the other two previous bounds to give a tighter game bound

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Future Directions

Future Direction

1. Our results rely on Zeckendorf decomposition properties. Generalize the known constant coefficient case into non-constant case. i.e. See what happens when generalizing Fibonacci sequence to the sequence $a(n+1) = n\, a(n) + a(n-1)$.
2. Construct the winning strategy for the 2nd player (in a 2 player game).
3. Construction of alliances with winning strategy in multiplayer game $(p > 2)$.
4. Further tighten the bound

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References I

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2. Li, R., Li, X., Miller, S. J., Mizgerd, C., Sun, C., Xia, D., & Zhou, Z.(2020). Deterministic zeckendorf games.*arXiv preprintarXiv:2006.16457.*