Lower-Order Biases in Elliptic Curve Fourier Coefficients

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Elliptic Curves

An *elliptic curve* E is the set of solutions (x, y) to an equation of the form

$$y^2 = x^3 + ax + b$$

with $a, b \in \mathbb{Z}$. For $p > 3$ the *elliptic curve Fourier coefficients* are

$$a_E(p) = p - \#\{(x, y) : y^2 \equiv x^3 + ax + b \mod p\}.$$

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The associated Dirichlet series

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}, \quad \Re(s) > \frac{3}{2}$$

can be analytically continued an L-function on all of $\mathbb{C}$.

Families and Moments

A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 = x^3 + A[T]x + B[T]$$

where $A[T], B[T]$ are polynomials in $\mathbb{Z}[T]$.

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- Each specialization of T to an integer t gives an elliptic curve $\mathcal{E}(t)$ over $\mathbb{Q}$.
- The r^{th} *moment* of the Fourier coefficients is

$$A_{r,\mathcal{E}}(p) = \sum_{t \bmod p} a_{\mathcal{E}(t)}(p)^r.$$

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Bias Conjecture

Second Moment Asymptotic [Michel]

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Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.

One Interpretation

Sato-Tate Law for Families without CM

For large primes p, the distribution of $\frac{a_{\mathcal{E}(t)}(p)}{\sqrt{p}}$, $t \in \{0, 1, \dots, p-1\}$, approaches a semicircle on $[-2, 2]$.

- In this case, the Bias Conjecture can be interpreted as approaching the semicircle second moment from below.

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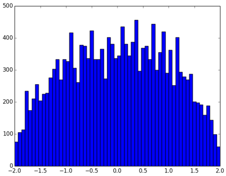

Figure: $a_{\mathcal{E}(t)}(p)$ for $y^2 = x^3 + Tx + 1$ at the 2014th prime

Implications for Excess Rank

- Katz-Sarnak's one-level density statistic is used to measure the average rank of curves over a family.
- More curves with rank than expected have been observed, though this excess average rank vanishes in the limit.
- Lower-order biases in the moments of families explain a small fraction of this excess rank phenomenon.

Negative Bias in the First Moment

The First Moment $A_{1,\mathcal{E}(t)}(p)$ and Family Rank [Rosen-Silverman]

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \frac{A_{1,E}(p)\log p}{p} = -rank(E(\mathbb{Q}[T]))$$

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- By the Prime Number Theorem, $A_{1,E}(p) = -rp + O(1)$ implies $rank(E(\mathbb{Q}[T])) = r$.
- We use this to study families of varying rank and understand the relationship between $A_{2,\mathcal{E}(t)}(p)$ and $rank(E(\mathbb{Q}[T]))$.

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Methods for Obtaining Explicit Formulas

For a family $\mathcal{E} : y^2 = x^3 + A[T]x + B[T]$, we can write

$$a_{\mathcal{E}(t)}(p) = -\sum_{x \mod p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol $\mod p$ given by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if } a^2 \equiv x \text{ for some } a \neq 0 \\ 0 & \text{if } x \equiv 0 \\ -1 & a^2 \not\equiv x \text{ for all } a \end{cases} \mod p$$

Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

$$\sum_{x \bmod p} \left(\frac{ax+b}{p}\right) = 0 \quad \text{if } p \nmid a$$

$$\sum_{x \bmod p} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid b^2 - 4ac \\ (p-1)\left(\frac{a}{p}\right) & \text{if } p \mid b^2 - 4ac \end{cases}$$

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Average Values of Legendre Symbols

The value of $\left(\frac{x}{p}\right)$ for $x \in \mathbb{Z}$, when averaged over all primes p, is 1 if x is a non-zero square, and 0 otherwise.

Rank 0 Families

Theorem [MMRW'14]: Rank 0 Families Obeying the Bias Conjecture

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bx + cT + d$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{a^2 - 3b}{p}\right)\right).$$

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- The average bias in the size p term is -2 or -1, according to whether $a^2 - 3b \in \mathbb{Z}$ is a non-zero square.

Families with Rank

Theorem [MMRW'14]: Families with Rank

For families of the form $\mathcal{E} : y^2 = x^3 + aT^2x + bT^2$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{-3a}{p}\right)\right) - \left(\sum_{x(p)} \left(\frac{x^3+ax}{p}\right)\right)^2$$

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- These include families of rank 0, 1, and 2.
- The average bias in the size p terms is -3 or -2, according to whether $-3a \in \mathbb{Z}$ is a non-zero square.

Families with Complex Multiplication

Theorem [MMRW'14]: Families with Complex Multiplication

For families of the form $\mathcal{E} : y^2 = x^3 + (aT + b)x$,

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- The average bias in the size p term is -1.
- The size p^2 term is not constant, but is on average p^2, and an analogous Bias Conjecture holds.

Families with Unusual Distributions of Signs

Theorem [MMRW'14]: Families with Unusual Signs

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 - (T+3)x + 1$,

$$A_{2,\mathcal{E}}(p) = p^2 - p\left(2 + 2\left(\frac{-3}{p}\right)\right) - 1.$$

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- The average bias in the size p term is -2.

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$$A_{2,\mathcal{E}}(p) = p^2 - p\left(2 + 2\left(\frac{-3}{p}\right)\right) - 1.$$

- The average bias in the size p term is -2.
- The family has an usual distribution of signs in the functional equations of the corresponding L-functions.

The Size $p^{3/2}$ Term

Theorem [MMRW'14]: Families with a Large Error

For families of the form
$\mathcal{E}: y^2 = x^3 + (T+a)x^2 + (bT + b^2 - ab + c)x - bc,$

$$A_{2,\mathcal{E}}(p) = p^2 - 3p - 1 + p \sum_{x \bmod p} \left(\frac{-cx(x+b)(bx-c)}{p} \right)$$

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- The size $p^{3/2}$ term is given by an elliptic curve coefficient and is thus on average 0.
- The average bias in the size p term is -3.

General Structure of the Lower Order Terms

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- have no size $p^{3/2}$ term or a size $p^{3/2}$ term that is on average 0;
- exhibit their negative bias in the size p term;
- be determined by polynomials in p, elliptic curve coefficients, and congruence classes of p (i.e. values of Legendre symbols).

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Numerical Methods

- As complexity of coefficients increases, it is much harder to find an explicit formula.
- We can always just calculate the second moment from the explicit formula; if $\mathcal{E}$: $y^2 = f(x)$, we have

$$A_{2,\mathcal{E}}(p) = \sum_{t(p)} \left(\sum_{x(p)} \left(\frac{f(x)}{p} \right) \right)^2 .$$

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- Takes an hour for the first 500 primes.
- Optimizations?

Numerical Methods

Consider the family $y^2 = f(x) = ax^3 + (bt + c)x^2 + (dt + e)x + f$. By similar arguments used to prove special cases,

$$A_{2,E}(p) = p^2 - 2p + pC_0(p) - pC_1(p) - 1 + \#_1,$$

where

$$C_0(p) = \sum_{x(p)} \sum_{y(p):\ \beta(x,y)\equiv 0} \left(\frac{A(x)A(y)}{p}\right),$$

$$C_1(p) = \sum_{x(p):\ \beta(x,x)\equiv 0} \left(\frac{A(x)^2}{p}\right),$$

$$\#_1 = p \sum_{x(p)} \sum_{y(p):\ A(x)\equiv 0 \text{ and } A(y)\equiv 0} \left(\frac{B(x)B(y)}{p}\right),$$

and β, A, and B are polynomials.

Numerical Methods

- $C_o(p)$ ordinarily $O(p^2)$ to compute.
- Sum over zeros of $\beta(x, y) \mod p$
- Fixing an x, β is a quadratic in y. So, with the quadratic formula mod p, we know where to look for y to see if there is a zero.
- Now $O(p)$; runs from 6000^{th} to 7000^{th} prime in an hour.

Potential Counterexamples

Families of Rank as Large as 3

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bT^2x + cT^2$ with $b, c \neq 0$, we can expand the second moment as

$$A_{2,\mathcal{E}}(p) = p^2 + p \sum_{P(x,y)\equiv 0} \left(\frac{(x^3+bx)(y^3+by)}{p} \right) + p \left[\sum_{x^3+bx\equiv 0} \left(\frac{ax^2+c}{p} \right) \right]^2 - p \sum_{P(x,x)\equiv 0} \left(\frac{x^3+bx}{p} \right)^2 - p \left(2 + \left(\frac{-b}{p} \right) \right) - \left[\sum_{x \bmod p} \left(\frac{x^3+bx}{p} \right) \right]^2 - 1$$

where $P(x,y) = bx^2y^2 + c(x^2+xy+y^2) + bc(x+y)$.

A Positive Size p Term?

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- The term $p\sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ is of size $p^{3/2}$.

Analyzing the Size $p^{3/2}$ Term

We averaged $\sum_{P(x,y)\equiv 0}\left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ over the first 10000 primes for several rank 3 families of the form $\mathcal{E}: y^2 = x^3 + ax^2 + bT^2x + cT^2$.

Family	Average
$y^2 = x^3 + 2x^2 - 4T^2x + T^2$	-0.0238
$y^2 = x^3 - 3x^2 - T^2x + 4T^2$	-0.0357
$y^2 = x^3 + 4x^2 - 4T^2x + 9T^2$	-0.0332
$y^2 = x^3 + 5x^2 - 9T^2x + 4T^2$	-0.0413
$y^2 = x^3 - 5x^2 - T^2x + 9T^2$	-0.0330
$y^2 = x^3 + 7x^2 - 9T^2x + T^2$	-0.0311

The Right Object to Study

$c_{3/2}(p) := \sum_{P(x,y)\equiv 0} \left(\frac{(x^3+bx)(y^3+by)}{p}\right)$ is not a natural object to study.

An example distribution for $y^2 = x^3 + 2x^3 - 4T^2x + T^2$.

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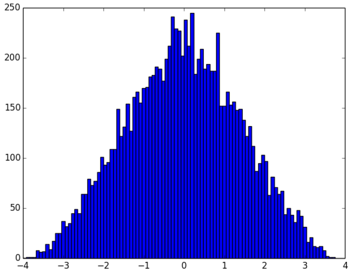

Figure: $c_{3/2}(p)$ over the first 10000 primes

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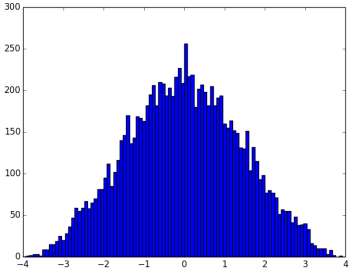

Figure: $-\sum_{x \bmod p} \left(\frac{x^3+x+1}{p}\right) - \sum_{x \bmod p} \left(\frac{x^3+x+2}{p}\right)$ over the first 10000 primes

More Error Distributions

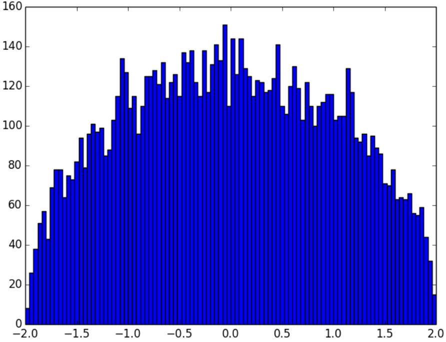

Figure: $c_{3/2}(p)$ over $y^2 = 4x^3 + 5x^2 + (4t-2)x + 1$

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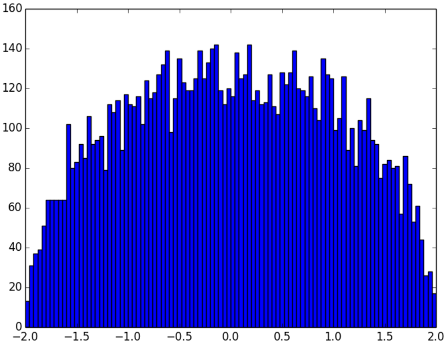

Figure: $-\sum_{x \bmod p}\left(\frac{x^3+x+1}{p}\right)$ distribution

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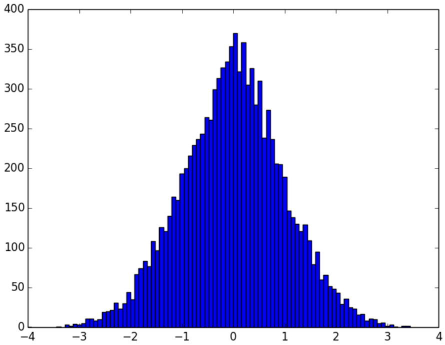

Figure: $c_{3/2}(p)$ over $y^2 = 4x^3 + (4t+1)x^2 + (-4t-18)x + 49$

More Error Distributions

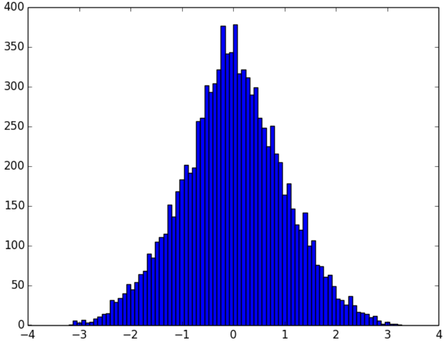

Figure: $-\sum_{x \bmod p} \left(\frac{x^5+x^3+x^2+x+1}{p}\right)$ distribution

Summary of $p^{3/2}$ Term Investigations

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- appear to be governed by (hyper)elliptic curve coefficients;
- may be hiding negative contributions of size p;
- prevent us from numerically measuring average biases that arise in the size p terms.

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- Does the average bias always occur in the terms of size p?
- Does the Bias Conjecture hold similarly for all higher even moments?
- What other (families of) objects obey the Bias Conjecture? Kloosterman sums? Cusp forms of a given weight and level? Higher genus curves?

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- SMALL REU 2014
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