Large gaps between zeros of GL(2) L-functions

Owen Barrett, Brian McDonald, Patrick Ryan

JOINT WITH: Steven J. Miller, Caroline Turnage-Butterbaugh, Karl Winsor

Gaps collaboration, SMALL PANTHers 2014, Williams College

Québec-Maine Number Theory Conference
Université Laval
September 27, 2014

Outline

Gaps between Critical Zeros

Results on Gaps and Shifted Second Moments

Methods of Proof

Extension to Maass Forms

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Philosophy: Critical-zero statistics of L-functions agree with eigenvalue statistics of large random matrices

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- Hejhal, Rudnick and Sarnak - Higher correlations of automorphic L-functions
- Odlyzko - further evidence through extensive numerical computations

Consecutive Zero Spacings

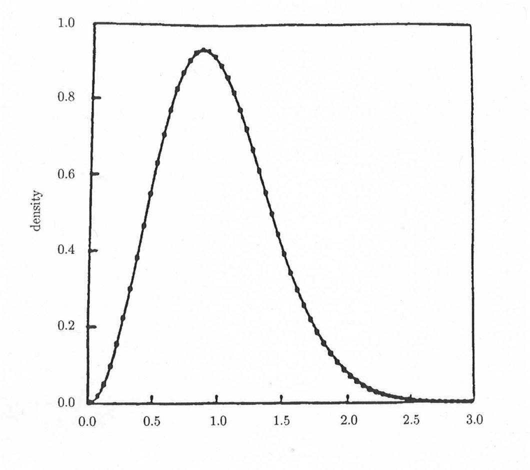

Figure: Consecutive zero spacings of $\zeta(s)$ versus GUE predictions

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$$\text{Letting} \quad \Lambda = \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}},$$

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- The best unconditional result for the Riemann zeta function is

$$\Lambda > 2.69.$$

Degree 2 Case

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Theorem 1 (Turnage-Butterbaugh '14).
Let $T \geqslant 2$ and $\varepsilon > 0$. Let $\zeta_K(s)$ be the Dedekind zeta function attached to a quadratic number field K with discriminant d satisfying $|d| \leqslant T^{\frac{7}{9}-\varepsilon}$. Let $\mathcal{S} := \{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $\zeta_K\left(\frac{1}{2}+it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}$. Then

$$\kappa_T \geqslant \sqrt{6}\frac{\pi}{\log\sqrt{|d|}T}\left(1+O(d^{\varepsilon}\log T^{-1})\right)$$

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- Assuming GRH, this means $\Lambda \geqslant \sqrt{6}$.

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A Lower Bound on Large Gaps

We proved the following unconditional theorem for an L-function associated to a holomorphic cusp form f.

Theorem 2 (BMMRTW '14).
Let $\mathcal{S} := \{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}$. Then

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right),$$

where c_f is the residue of the Rankin-Selberg convolution $L(s, f \times \bar{f})$ at $s = 1$.

Interpretation

If we assume GRH for interpretive purposes, there are infinitely many normalized gaps between consecutive zeros at least $\sqrt{3}$ times the mean spacing, i.e.,

$$\Lambda \geqslant \sqrt{3}.$$

Shifted Moment Result

In order to prove our theorem, we use a method due to R.R. Hall, along with the following shifted moment result.

Theorem 3 (BMMRTW '14).

$$\int_T^{2T} L\left(\frac{1}{2}+it+\alpha, f\right) L\left(\frac{1}{2}-it+\beta, f\right) dt$$
$$= c_f T \sum_{n\geqslant 0} \frac{(-1)^n 2^{n+1}(\alpha+\beta)^n (\log T)^{n+1}}{(n+1)!} + O\left(T(\log T)^{1-\delta}\right),$$

where $\alpha, \beta \in \mathbf{C}$ and $|\alpha|, |\beta| \ll 1/\log T$.

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Shifted Moments Proof Technique

- Following a method due to Ramachandra, we consider

$$L(s+\alpha, f) = \sum_{n \geqslant 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n \leqslant X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s),$$

where $\lambda_f(n)$ are the Fourier coefficients of $L(s, f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.

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  where $\lambda_f(n)$ are the Fourier coefficients of $L(s, f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.
- We have an analogous expression for $L(1-s+\beta, f)$.

Shifted Moments Proof Technique

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where each factor gives rise to four products, resulting in sixteen total products to estimate.

- Using a generalization of Montgomery and Vaughan's mean value theorem and contour integration we are able to estimate this product and compute the resulting moments.

Shifted Moment Result for Derivatives

- The shifted moment result allows us to deduce lower order terms and moments of derivatives of L-functions by means of differentiation and Cauchy's integral formula.

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- The shifted moment result allows us to deduce lower order terms and moments of derivatives of L-functions by means of differentiation and Cauchy's integral formula.
- We derive an expression for

$$\int_T^{2T} L^{(\mu)}\left(\frac{1}{2}+it, f\right) L^{(\nu)}\left(\frac{1}{2}-it, f\right)\,dt,$$

where $T \geqslant 2$ and $\mu, \nu \in \mathbf{Z}^+$. We use this result in Hall's method to obtain the lower bound stated in our theorem.

Modified Wirtinger Inequality

Using Hall's method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg.

Lemma 1 (Bredberg).

Let $y : [a, b] \to \mathbf{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$\int_a^b |y(x)|^2 dx \leqslant \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 dx.$$

Proving our Result

- We define the function
$$g(t) := e^{i\rho t \log T} L\left(\tfrac{1}{2} + it, f\right),$$
where ρ is a real parameter to be chosen later. We fix f and let $\tilde{\gamma}$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.

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  where ρ is a real parameter to be chosen later. We fix f and let $\tilde{\gamma}$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.
- The crucial property of this function is that it has the same zeroes as $L(s, f)$, namely $g(t) = 0$ when $t = \tilde{\gamma}$. We use this function in the modified Wirtinger's inequality.

Proving our Result

- We apply sub-convexity bounds along the critical line to establish:

$$\int_T^{2T} |g(t)|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 dt + O\left(T^{\frac{2}{3}} (\log T)^{\frac{5}{6}}\right).$$

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- Noting that our $g(t)$ and $g'(t)$ may be expressed in terms of $L\left(\frac{1}{2} + it, f\right)$ and its derivatives, we can write our inequality explicitly in terms of formulæ given by our mixed moment theorem.

Finishing the Proof

- After substituting our formulæ, we have the inequality:

$$\frac{\kappa_T^2}{\pi^2} \geqslant \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left(1 + O(\log T)^{-\delta}\right).$$

Finishing the Proof

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- The polynomial in ρ finds its minimum at $\rho = 1$, yielding the result

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right).$$

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Hypotheses

The success of these methods only depends on a few key properties of the L-function in question. For f a primitive form on $GL(2)$ over $\mathbf{Q}$ given by a Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \Re(s) > 1,$$

we expect the same lower bound on large gaps of $\sqrt{3}$ given the following assumptions:

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we expect the same lower bound on large gaps of $\sqrt{3}$ given the following assumptions:

- $L(s,f)$ has an analytic continuation to an entire function of order 1.
- $L(s,f)$ satisfies a function equation of the form

  $$\Lambda(s,f):=L(s,f_\infty)L(s,f)=\epsilon_f\Lambda(1-s,\bar{f})$$

  where

  $$L(s,f_\infty)=Q^s\Gamma\left(\frac{s}{2}+\mu_1\right)\Gamma\left(\frac{s}{2}+\mu_2\right).$$

Hypotheses

- The Rankin-Selberg convolution $L(s, f \times \bar{f})$, given by the Dirichlet series

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- For some small $\delta > 0$, we have a subconvexity bound

$$\left| L\left(\frac{1}{2} + it, f\right) \right| \ll |t|^{\frac{1}{2} - \delta}.$$

Hypotheses

- Michel and Venkatesh proved a subconvexity bound for primitive $\mathrm{GL}(2)$ L-functions over $\mathbf{Q}$.

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- All hypotheses are known for holomorphic cusp form L-functions.
- For L-functions associated to Maass forms, some are conjectural.

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We would like extend our deepest thanks to our advisors, Caroline Turnage-Butterbaugh and Steven J. Miller.

We would also like to acknowledge the support of NSF Grant DMS1347804 and Williams College.

- Owen Barrett – *owen.barrett@yale.edu*
- Brian McDonald – *bmcdon11@u.rochester.edu*
- Steven J. Miller – *sjm1@williams.edu*
- Patrick Ryan – *patrickryan01@college.harvard.edu*
- Caroline Turnage-Butterbaugh – *cturnagebutterbaugh@gmail.com*
- Karl Winsor – *krlwnsr@umich.edu*