Progress Report: Statistics over function fields

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Sage Days 21: Function Fields
Seattle, May 26, 2010

Summary

Goal: Use ellff library to investigate statistics of zeros of elliptic curve L-functions in function field.

Successfully installed library on Miller's laptop.

Gathered data on ranks and first zero above the central point.

Plan: add additional functions to ellff library.

Sage Code using ellff (which is not yet finalized)

```
import sage.libs.ellff as ellff
R = ZZ['T']
R.inject_variables()

def test(p=5):F=GF(p)
R.<t>= F['t']
K=Frac(R)
return ellff.ellff_EllipticCurve(K,[0,-1-t,0,t,0])

def twist(E,f,tables=False,force=False,verbose=False):E_twist=
 E.quadratic_twist(f,tables=tables,force=force,verbose=verbose)

if verbose:print"finite bad reduction:"
print"    M_sp :",E_twist.__finite_M_sp
print"    M_ns :",E_twist.__finite_M_ns
print"     A   :",E_twist.__finite_A

print
print"refined finite additive reduction:"
print"    I^*     :",E_twist.__finite_I_star
print"   II,II^*  :",E_twist.__finite_II,E_twist.__finite_II_star
print"  III,III^* :",E_twist.__finite_III,E_twist.__finite_III_star
print"   IV,IV^*  :",E_twist.__finite_IV,E_twist.__finite_IV_star

return E_twist

def pullback(E,f,tables=False,force=False,verbose=False):E_pullback=
 E.pullback(f,tables=tables,force=force,verbose=verbose)

if verbose:print"finite bad reduction:"
print"    M_sp :",E_pullback.__finite_M_sp
print"    M_ns :",E_pullback.__finite_M_ns
print"     A   :",E_pullback.__finite_A

print
print"refined finite additive reduction:"
print"    I^*     :",E_pullback.__finite_I_star
print"   II,II^*  :",E_pullback.__finite_II,E_pullback.__finite_II_star
print"  III,III^* :",E_pullback.__finite_III,E_pullback.__finite_III_star
print"   IV,IV^*  :",E_pullback.__finite_IV,E_pullback.__finite_IV_star

return E_pullback
```

Sage Code using ellff (which is not yet finalized)

```
p = 5
R.<t> = GF(p)['t']
R.inject_variables()
print

E = test(p)

print "finite bad reduction:"
print "    M_sp : ", E.__finite_M_sp
print "    M_ns : ", E.__finite_M_ns
print "     A   : ", E.__finite_A
print
print "L-fcn = ", E.L_function()
```

Sage Code using ellff (which is not yet finalized)

```
# CREATE DATA

data_list = [[p, E.a4, E.a6]]
data_vec = [[p, E.a4, E.a6]]
print "Printing information on our initial elliptic curve"
print E
print "Prime is ", p
Edisc = 4 * (E.a4)^3 + 27 * (E.a6)^2
print "a4 = ", E.a4, " a6 = ", E.a6, " and disc = ", Edisc
for a in range(p): for b in range(p): for c in range(p): for d in range(p): for e in range(p): f =
 a + b * t + c * t^2 + d * t^3 + e * t^4 + t^5
rk = 0
args = []
args_list = []
# print (f, Edisc, gcd(f, Edisc))
if gcd(f, diff(f)).degree() == 0 and gcd(f, Edisc).degree() == 0: E_twist =
 twist(E, f, tables = True, force = True)
L = E_twist.L_function()
# print "f = ", f, " : ", L, ", ", factor(L), ", ", "HERE"
for pi, ex in list(L.factor()): v = []
# print "ex = ", ex, ", pi = ", pi, ", pi.roots() = ", pi.roots(CDF),
for r, m in pi.roots(CDF): # print "r = ", r, ", m = ", m
```

Sage Code using ellff (which is not yet finalized)

```
assert m == 1
v.append(r.arg())
for i in range(ex): if r.arg() == 0: rk = rk + 1
args_list.append(r.arg())
args.append([ex, v])
#print "v = ", v
#build the data file consisting of the twist,
        L-function, #sign of the f.e., the rank, and the zeroes
#data_vec stores a zero with its multiplicities as a vector
#data_list stores a zero as many times as its multiplicity
data_vec.append([f, L, E_twist.sign, rk, args])
data_list.append([f, L, E_twist.sign, rk, args_list])

        for i in range(len(data_list)-1):
         if data_list[i+1][1].degree() ≠ data_list[2][1].degree():
          raise ValueError("Degree of L-function at %s is %s" % (i+1, data_list[i+1][1]))
if data_list[i+1][1].degree() ≠ len(data_list[i+1][4]):
          raise ValueError("Not enough zeros found at %s" % (i+1))

        data_str = str(data_list).replace('[', '{').replace(']', '}')

fname = str(E.a4) + "-" + str(E.a6) + ".dat"
file = open(fname, "w")
file.write(data_str)
file.close()
```

Rank of degree 4 twists

$E : y^2 = x^3 + (3 + 2t + 3t^2)x + (4 + 4t + 4t^2 + 4t^3)$.

Twisting by square-free $a + bt + ct^2 + dt^3 + t^4$ relatively prime to discriminant.

Data incomplete for $p = 11$

	$p = 5$ (345)	$p = 7$ (1573)	$p = 11$ (5000)
Rank 0	39.13	41.96	42.53
Rank 1	51.01	50.16	49.86
Rank 2	9.86	7.69	7.33
Rank 3	0.00	0.19	0.27
Rank 4	0.00	0.00	0.02

Rank of degree 5 twists

$E : y^2 = x^3 + (3 + 2t + 3t^2)x + (4 + 4t + 4t^2 + 4t^3)$.

Twisting by square-free $a + bt + ct^2 + dt^3 + t^4$ relatively prime to discriminant.

Data incomplete for $p = 7$ (roughly 50% done?)

	$p = 5$ (1739)	$p = 7$ (7530)
Rank 0	44.97	44.22
Rank 1	49.86	49.53
Rank 2	4.48	5.83
Rank 3	0.35	0.41
Rank 4	0.00	0.00

First normalized zero above central point

$E : y^2 = x^3 + (3 + 2t + 3t^2)x + (4 + 4t + 4t^2 + 4t^3)$.

Twisting by square-free $a + bt + ct^2 + dt^3 + t^4$ relatively prime to discriminant.

Data incomplete for $p = 11$

	$p = 5$ (345)	$p = 7$ (1573)	$p = 11$ (5000)
mean Rank 0	.458	.437	.432
mean all even	.367	.369	.368

First normalized eigenangle above 0: 23,040 SO(4) matrices: Mean = .357; 23,040 SO(6) matrices: Mean = .325, $N \to \infty$ scaling limit: Mean = .321.

First normalized zero above central point

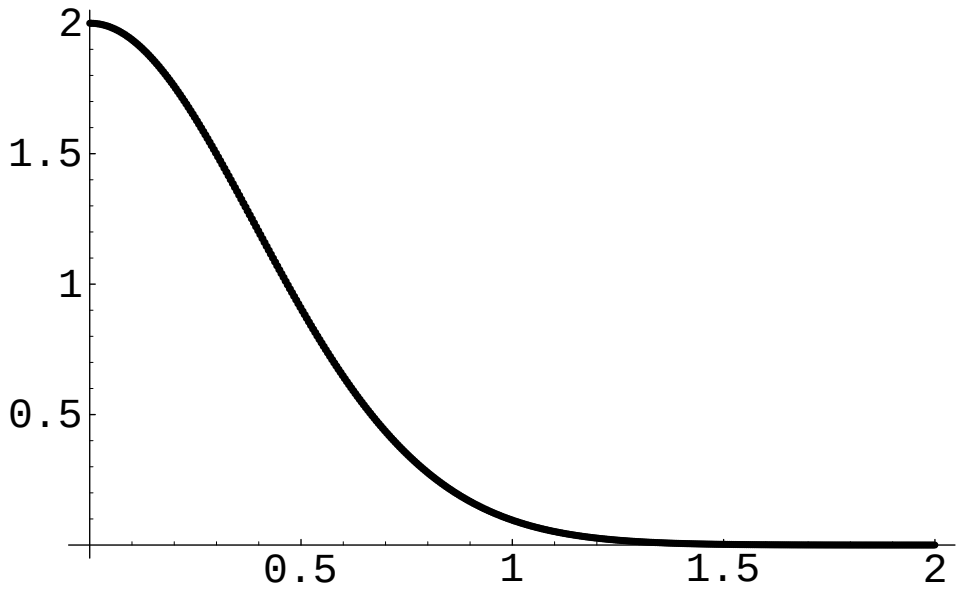

Figure: First zero for $N \to \infty$ limit of SO($2N$).

Histograms for first zero (same curve and degree twists as above)

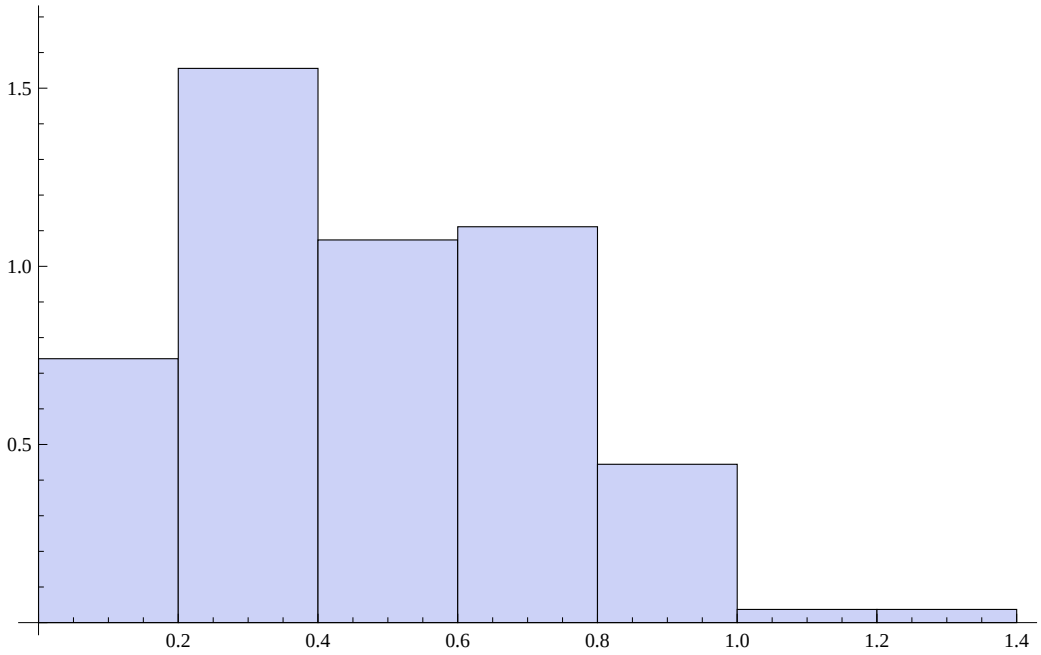

Figure: First zero for $p = 5$ rank 0 curves: Mean = .458

Histograms for first zero (same curve and degree twists as above)

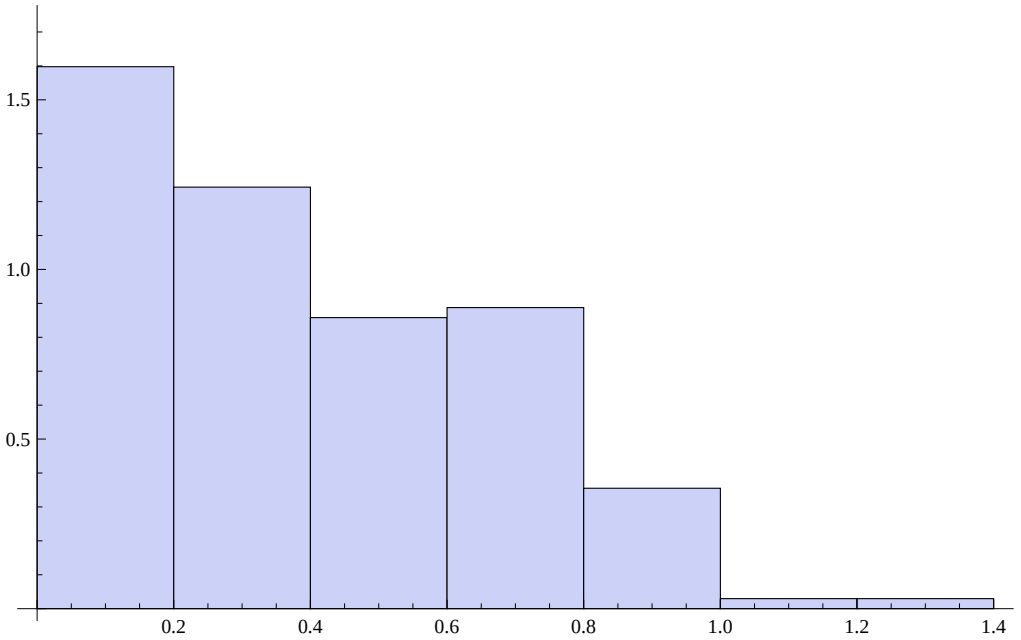

Figure: First zero for $p = 5$ rank even curves: Mean = .367

Histograms for first zero (same curve and degree twists as above)

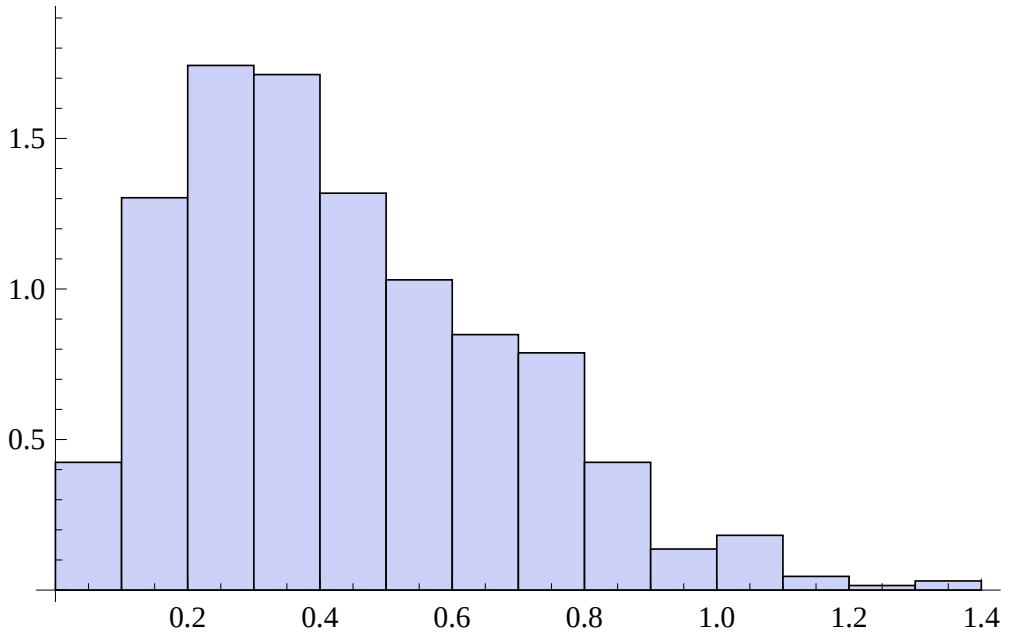

Figure: First zero for $p = 7$ rank 0 curves: Mean = .437

Histograms for first zero (same curve and degree twists as above)

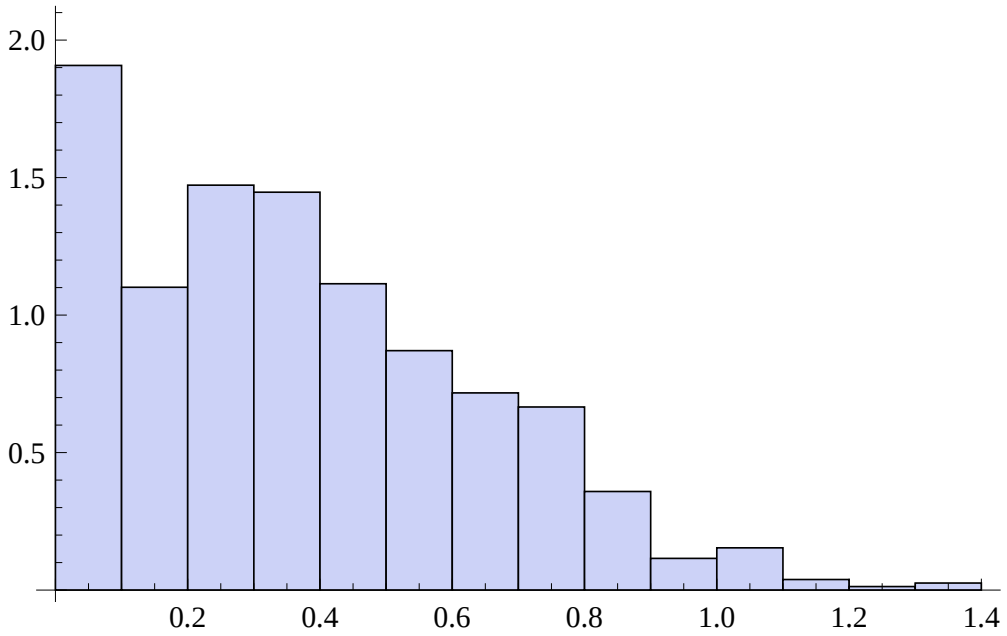

Figure: First zero for $p = 7$ rank even curves: Mean = .369

Histograms for first zero (same curve and degree twists as above)

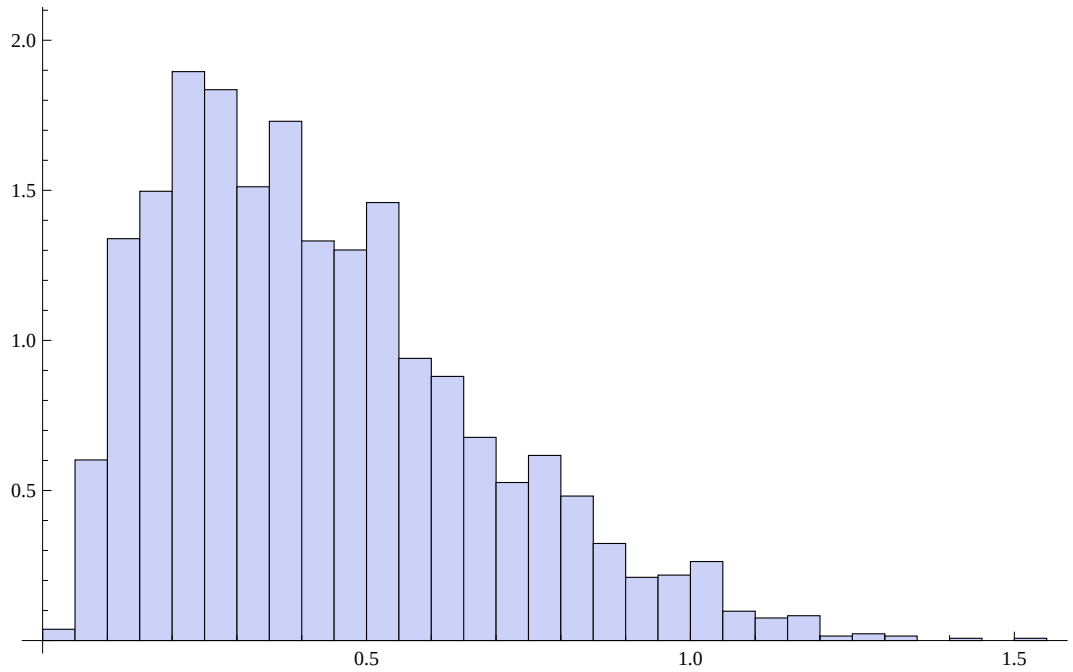

Figure: First zero for $p = 11$ rank 0 curves: Mean = .432

Histograms for first zero (same curve and degree twists as above)

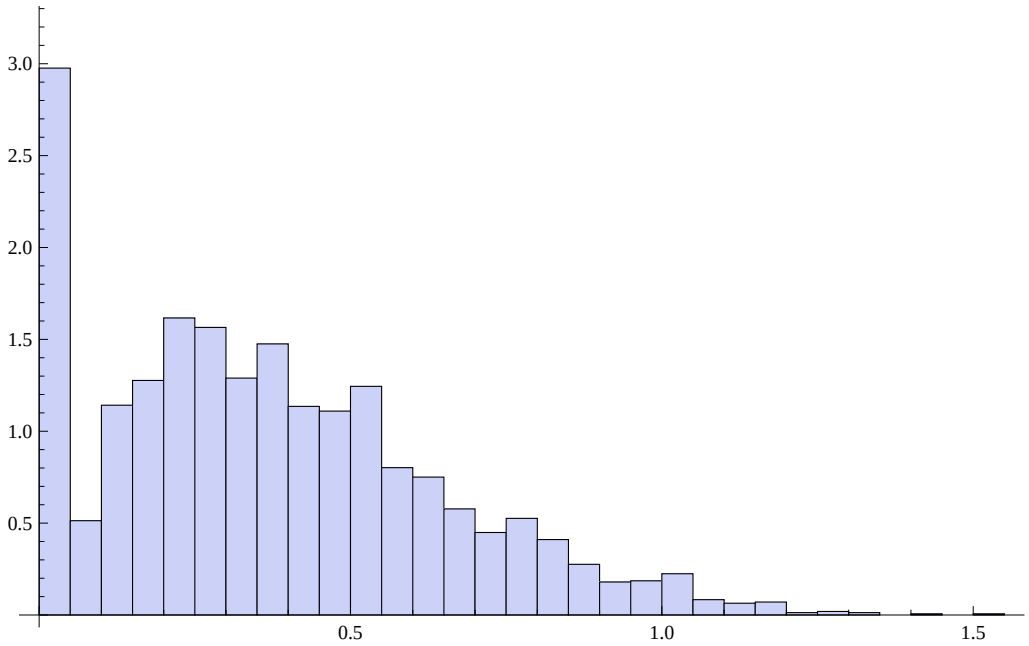

Figure: First zero for $p = 11$ rank even curves: Mean = .368