Distribution of Summands in Generalized Zeckendorf Decompositions

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Introduction
Goals of the Talk

- Generalize Zeckendorf decompositions
- Analyze average gap distribution
- Analyze distribution of individual gaps
Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$. 
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**Lekkerkerkerker’s Theorem (1952)**

The average number of summands in the Zeckendorf decomposition for integers in \( [F_n, F_{n+1}) \) tends to \( \frac{n}{\varphi^2 + 1} \approx 0.276n \),
where \( \varphi = \frac{1+\sqrt{5}}{2} \) is the golden mean.
Central Limit Type Theorem [KKMW]

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

Figure: Number of summands in $[F_{2010}, F_{2011})$. 
Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

\[ H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \quad n \geq L \]

with \( H_1 = 1, \quad H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \quad n < L, \)
coefficients \( c_i \geq 0; \quad c_1, c_L > 0 \) if \( L \geq 2; \quad c_1 > 1 \) if \( L = 1. \)
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- **Zeckendorf**
- **Lekkerkerker**: Average number summands is \( C_{\text{Lek}} n + d. \)
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- Central Limit Type Theorem
Gaps Between Summands
Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1.$$
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Example: For $H_1 + H_8 + H_{18}$, the gaps are 7 and 10.
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**Definition**

Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length $m$. 
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**Definition**

Let \( P_n(m) \) be the probability that a gap for a decomposition in \([H_n, H_{n+1})\) is of length \( m \).

**Big Question:** What is \( P(m) = \lim_{n \to \infty} P_n(m) \)?
Main Results

Theorem (Base $B$ Gap Distribution (SMALL 2011))

For base $B$ decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = \frac{1}{\phi^k}$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.
Main Results

**Theorem (Distribution of Bulk Gaps (SMALL 2012))**

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length $L$ where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

\[
P(j) = \begin{cases} 
1 - (\frac{a_1}{C_{Lek}})(2\lambda_1^{-1} + a_1^{-1} - 3) & : j = 0 \\
\lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & : j = 1 \\
(\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & : j \geq 2
\end{cases}
\]
Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker ⇒ total number of gaps \( \sim F_{n-1} \frac{n}{\phi^2 + 1} \).
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Let \( X_{i,j} = \#\{m \in [F_n, F_{n+1}): \text{decomposition of } m \text{ includes } F_i, F_j, \text{ but not } F_q \text{ for } i < q < j\} \).
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\[
P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.
\]
Calculating $X_{i,i+k}$

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For the indices less than $i$: $F_{i-1}$ choices. Why? Have $F_i$ as largest summand and follows by Zeckendorf:

$\#(F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.
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For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \ldots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand $F_{n-k-i}$. 
Calculating $X_{i,i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

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So total choices number of choices is $F_{n-k-2-i}F_{i-1}$. 
Determining $P(k)$

Recall,

$$P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.$$ 

Use Binet’s formula and get sums of geometric series. Then $P(k) = 1/\phi^k$.

**Figure:** Distribution of summands in $[F_{1000}, F_{1001})$. 
Individual Gaps
**Decomposition:** \( m = \sum_{j=1}^{k(m)} F_{ij} \)
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Individual gap measure:
\[ \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta (x - (i_j - i_{j-1})) \]
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\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta \left( x - (i_j - i_{j-1}) \right)
\]

**Theorem (Distribution of Individual Gaps (SMALL 2012))**

Gap measures \( \nu_{m;n} \) converge almost surely to average gap measure.
**Proof Sketch of Individual Gaps**

\[ \mu_{m,n}(t) = \int x^t d\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t. \]
Proof Sketch of Individual Gaps

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- Show \( \mathbb{E}_m[\mu_{m,n}(t)] \) equals average gap moments, \( \mu(t) \).
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- Show \( \mathbb{E}_m[\mu_{m,n}(t)] \) equals average gap moments, \( \mu(t) \).

- Show \( \mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^2] \) and \( \mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^4] \) tend to zero.
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- \( \mu_{m,n}(t) = \int x^t d\nu_{m;n}(x) = \frac{1}{k(m) - 1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t. \)

- Show \( E_m[\mu_{m;n}(t)] \) equals average gap moments, \( \mu(t) \).

- Show \( E_m[(\mu_{m;n}(t) - \mu(t))^2] \) and \( E_m[(\mu_{m;n}(t) - \mu(t))^4] \) tend to zero.

**Key ideas:** (1) Replace \( k(m) \) with average (Gaussianity); (2) use \( X_{i,i+g_1,j,j+g_2} \).
Future Research
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- Extend to recurrences with coefficients that can be zero.
- Generalize to signed decompositions
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