Distribution of Summands in Generalized Zeckendorf Decompositions

Amanda Bower and Rachel Insoft
Joint with: Olivia Beckwith, Louis Gaudet, Shiyu Li, and Steven J. Miller, Philip Tosteson

MAA General Contributed Paper Session: Research in Number Theory, I
2013 Joint Math Meetings
San Diego, CA, January 9, 2013

http://web.williams.edu/Mathematics/sjmiller/public_html/jmm2013.html

Introduction

Goals of the Talk

- Generalize Zeckendorf decompositions
- Analyze average gap distribution
- Analyze distribution of individual gaps

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5, \dots$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1,\ F_2 = 2,\ F_3 = 3,\ F_4 = 5, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4$.

Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Central Limit Type Theorem [KKMW]

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

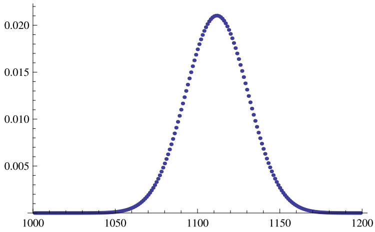


Figure: Number of summands in $[F_{2010}, F_{2011})$.

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \; n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \; n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf
- Lekkerkerker: Average number summands is $C_{\mathrm{Lek}} n + d$.

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf
- Lekkerkerker: Average number summands is $C_{\mathrm{Lek}} n + d$.
- Central Limit Type Theorem

Gaps Between Summands

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1},\ i_{n-1} - i_{n-2},\ \ldots\ ,\ i_2 - i_1.$$

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1},\ i_{n-1} - i_{n-2},\ \ldots\ ,\ i_2 - i_1.$$

Example: For $H_1 + H_8 + H_{18}$, the gaps are 7 and 10.

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1},\ i_{n-1} - i_{n-2},\ \ldots,\ i_2 - i_1.$$

Example: For $H_1 + H_8 + H_{18}$, the gaps are 7 and 10.

Definition

Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length m.

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1},\ i_{n-1} - i_{n-2},\ \dots,\ i_2 - i_1.$$

Example: For $H_1 + H_8 + H_{18}$, the gaps are 7 and 10.

Definition

Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length m.

Big Question: What is $P(m) = \lim_{n\to\infty} P_n(m)$?

Main Results

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Main Results

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ *be a positive linear recurrence of length* L *where* $c_i \geq 1$ *for all* $1 \leq i \leq L$. *Then* $P(j) =$

$$\begin{cases} 1 - \left(\frac{a_1}{C_{Lek}}\right)(2\lambda_1^{-1} + a_1^{-1} - 3) & : j = 0 \\ \lambda_1^{-1}\left(\frac{1}{C_{Lek}}\right)(\lambda_1(1 - 2a_1) + a_1) & : j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & : j \geq 2 \end{cases}$$

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Let $X_{i,j} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_j, but not F_q for $i < q < j\}$.

Proof of Bulk Gaps for Fibonacci Sequence

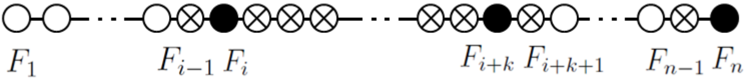
Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Let $X_{i,j} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_j, but not F_q for $i < q < j\}$.

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

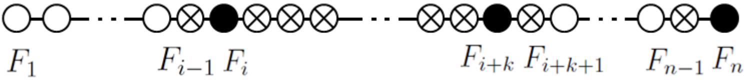
Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?



Calculating $X_{i,i+k}$

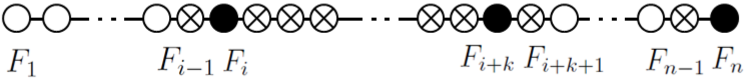
How many decompositions contain a gap from F_i to F_{i+k}?



For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?

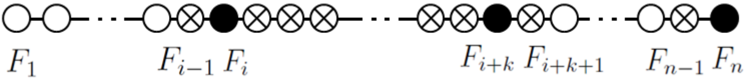


For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand F_{n-k-i}.

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?



For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand F_{n-k-i}.

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

Recall,

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Use Binet's formula and get sums of geometric series. Then $P(k) = 1/\phi^k$.

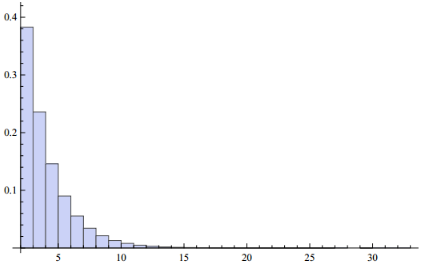


Figure: Distribution of summands in $[F_{1000}, F_{1001})$.

Individual Gaps

- Decomposition: $m = \sum_{j=1}^{k(m)} F_{i_j}$

- Decomposition: $m = \sum_{j=1}^{k(m)} F_{i_j}$

- Individual gap measure:
  $\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right)$

- Decomposition: $m = \sum_{j=1}^{k(m)} F_{i_j}$
- Individual gap measure:
  $\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right)$

Theorem (Distribution of Individual Gaps (SMALL 2012))

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure.

Proof Sketch of Individual Gaps

- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t.$

Proof Sketch of Individual Gaps

- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.

Proof Sketch of Individual Gaps

- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.
- Show $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^4]$ tend to zero.

Proof Sketch of Individual Gaps

- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.
- Show $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^4]$ tend to zero.

Key ideas: (1) Replace $k(m)$ with average (Gaussianity); (2) use $X_{i,i+g_1,j,j+g_2}$.

Future Research

Future Research

Future Research

- Extend to recurrences with coefficients that can be zero.
- Generalize to signed decompositions

Acknowledgements

Thanks to...

- NSF Grant DMS0850577
- NSF Grant DMS0970067
- MAA
- Our peers at Williams SMALL REU Summers 2010, 2011, and 2012

References

References

References

- Beckwith, Bower, Gaudet, Insoft, Li, Miller and Tosteson: Bulk gaps for average gap measure.
  http://arxiv.org/abs/1208.5820
- Kologlu, Kopp, Miller and Wang: Gaussianity for Fibonacci case.
  http://arxiv.org/pdf/1008.3204
- Miller - Wang: Gaussianity in general.
  http://arxiv.org/pdf/1008.3202
- Miller - Wang: Survey paper.
  http://arxiv.org/pdf/1107.2718