Low-lying zeros of GL(2) $L$-functions

Steven J Miller
Williams College

Steven.J.Miller@williams.edu


Maass forms joint with Levent Alpoge, holomorphic cusp forms with Chris Hughes, Geoff Iyer and Nicholas Triantafillou, elliptic curves with Eduardo Dueñez, Duc Khim Huynh, Jon Keating and Nina Snaith

AMS Special Session on Arithmetic Statistics, I
Room 31A, Upper Level, San Diego
Thursday January 10, 2013, 8:00am
Introduction
Summary

- Quick review of Katz-Sarnak philosophy.

- Symmetry type of compound families of $L$-functions.

- New model (excised orthogonal ensemble) for zeros of elliptic curve $L$-functions near the central point for finite conductors.

- Increasing support for $n$-level densities of holomorphic cuspidal newforms: agree up to $\left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$ (previous was $\frac{1}{n-1}$).

- 1-level densities of level 1 Maass forms: up to $\left(-\frac{4}{3}, \frac{4}{3}\right)$ (in progress, possibly larger when incorporate non-trivial Kloosterman results).
Measures of Spacings: \( n \)-Level Density and Families

\[ \phi(x) := \prod_i \phi_i(x_i), \phi_i \text{ even Schwartz functions whose Fourier Transforms are compactly supported.} \]

\( n \)-level density

\[ D_{n,f}(\phi) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_1 \left( L_f \gamma_f^{(j_1)} \right) \cdots \phi_n \left( L_f \gamma_f^{(j_n)} \right) \]
Measures of Spacings: \( n \)-Level Density and Families

\[ \phi(x) := \prod_i \phi_i(x_i), \ \phi_i \text{ even Schwartz functions whose Fourier Transforms are compactly supported.} \]

\[ D_{n,f}(\phi) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_1 \left( L_{f \gamma_f}^{(j_1)} \right) \cdots \phi_n \left( L_{f \gamma_f}^{(j_n)} \right) \]

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).
Measures of Spacings: \( n \)-Level Density and Families

\[
\phi(x) := \prod_i \phi_i(x_i), \quad \phi_i \text{ even Schwartz functions whose Fourier Transforms are compactly supported.}
\]

\( n \)-level density

\[
D_{n,f}(\phi) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_1 \left( L_{f \gamma_f}^{(j_1)} \right) \cdots \phi_n \left( L_{f \gamma_f}^{(j_n)} \right)
\]

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

Katz-Sarnak Conjecture

For a ‘nice’ family of \( L \)-functions, the \( n \)-level density depends only on a symmetry group attached to the family.
Normalization of Zeros

Local (hard, use $C_f$) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: $\phi$ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n} \prod_{i} \phi_i \left( \frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n, \mathcal{G}(\mathcal{F})}(x) dx.$$
1-Level Densities

The Fourier Transforms for the 1-level densities are

\[
\begin{align*}
\mathcal{W}_{1,SO(\text{even})}(u) &= \delta_0(u) + \frac{1}{2} \eta(u) \\
\mathcal{W}_{1,SO}(u) &= \delta_0(u) + \frac{1}{2} \\
\mathcal{W}_{1,SO(\text{odd})}(u) &= \delta_0(u) - \frac{1}{2} \eta(u) + 1 \\
\mathcal{W}_{1,Sp}(u) &= \delta_0(u) - \frac{1}{2} \eta(u) \\
\mathcal{W}_{1,U}(u) &= \delta_0(u)
\end{align*}
\]

where \(\delta_0(u)\) is the Dirac Delta functional and

\[
\eta(u) = \begin{cases} 
1 & \text{if } |u| < 1 \\
\frac{1}{2} & \text{if } |u| = 1 \\
0 & \text{if } |u| > 1 
\end{cases}
\]
Some Number Theory Results

- **Orthogonal**: Iwaniec-Luo-Sarnak; Ricotta-Royer; ...: 1-level density for holomorphic even weight $k$ cuspidal newforms of square-free level $N$ (SO(even) and SO(odd) if split by sign); Miller; Young; ...: One and two-parameter families of elliptic curves.

- **Symplectic**: Rubinstein; Gao; Levinson-Miller; Entin, Roddity-Gershon and Rudnick; Ozluk-Snyder; ...: $n$-level densities for twists $L(s, \chi_d)$ of the zeta-function.

- **Unitary**: Fiorilli-Miller; Hughes-Rudnick; ...: Families of Primitive Dirichlet Characters.
Applications of $n$-level density

Order of vanishing at the central point.

Average rank $\cdot \phi(0) \leq \int \phi(x) W_{G(F)}(x) dx$ if $\phi$ non-negative.
Applications of $n$-level density

Order of vanishing at the central point.
Average rank · $\phi(0) \leq \int \phi(x) W_{G(F)}(x) dx$ if $\phi$ non-negative.

Bound percentage vanish to order $r$:

**Theorem (Miller, Hughes-Miller)**

*Using $n$-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any $r$ there is a $c_r$ such that probability of at least $r$ zeros at the central point is at most $c_n r^{-n}$.*

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$. 
Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.

- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.

- How to identify symmetry group in general? One possibility is by the signs of the functional equation:

- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise $\text{SO}(\text{even})$. (False!)
Explicit Formula

- $\pi$: cuspidal automorphic representation on $\text{GL}_n$.
- $Q_\pi > 0$: analytic conductor of $L(s, \pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi, j}$.
- Satake parameters $\{\alpha_{\pi, i}(p)\}_{i=1}^n$;
  $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi, i}(p)^\nu$.
- $L(s, \pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi, i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi, j} \frac{\log Q_\pi}{2\pi}\right) = \hat{g}(0) - 2 \sum_{p, \nu} \hat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right) \frac{\lambda_\pi(p^\nu) \log p}{p^{\nu/2} \log Q_\pi}$$
Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: \( c_L = 0 \) (resp, 1 or -1) if family \( L \) has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for \( \pi_1, p \times \pi_2, p \) are

\[
\left\{ \alpha_{\pi_1 \times \pi_2}(k) \right\}_{k=1}^{nm} = \left\{ \alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j) \right\}_{1 \leq i \leq n, 1 \leq j \leq m}.
\]

**Theorem (Dueñez-Miller)**

If \( \mathcal{F} \) and \( \mathcal{G} \) are *nice* families of \( L \)-functions, then

\[
c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}.
\]

Breaks analysis of compound families into simple ones.
### 1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) = -2 \sum_p \hat{g} \left( \frac{\log p}{\log R} \right) \frac{\log p}{\sqrt{p} \log R} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p) \right]$$

$$S_2(\mathcal{F}) = -2 \sum_p \hat{g} \left( 2 \frac{\log p}{\log R} \right) \frac{\log p}{p \log R} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) \right]$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) = c_\mathcal{F} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$
Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.

- First moment zero save for families of elliptic curves.

- Higher moments control convergence and can depend on arithmetic of family.
Elliptic Curves: First Zero Above Central Point
(With E. Dueñez, D. K. Huynh, J. P. Keating, N. Snaith)
Theoretical results

**Theorem: M– ’04**

For small support, one-param family of rank $r$ over $\mathbb{Q}(T)$:

$$
\lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi \left( \frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j} \right) = \int \varphi(x) \rho_G(x) dx + r \varphi(0)
$$

where $\mathcal{G} = \begin{cases} 
SO(\text{odd}) & \text{if half odd} \\
SO(\text{even}) & \text{if all even} \\
\text{weighted average} & \text{otherwise.}
\end{cases}$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

**Independent Model:**

$$
\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r} & 2r \\ 2r & g \end{pmatrix} : g \in SO(2N - 2r) \right\}.
$$
RMT: Theoretical Results ($N \to \infty$)

1st normalized evalue above 1: $SO(\text{even})$
RMT: Theoretical Results ($N \to \infty$)

1st normalized value above 1: SO(odd)
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

Figure 4a: 209 rank 0 curves from 14 rank 0 families, log(\text{cond}) \in [3.26, 9.98], median = 1.35, mean = 1.36
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median = .81, mean = .86.
Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.

- As the conductors increased, the repulsion decreased.

- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).
Spacings b/w Norm Zeros: Rank 0 One-Param Families over \( \mathbb{Q}(T) \)

- All curves have \( \log(\text{cond}) \in [15, 16] \);
- \( z_j = \) imaginary part of the \( j \)th normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over \( \mathbb{Q}(T) \);
- 701 rank 2 curves from the 21 one-param families of rank 0 over \( \mathbb{Q}(T) \).

<table>
<thead>
<tr>
<th></th>
<th>863 Rank 0 Curves</th>
<th>701 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median</strong> ( z_2 - z_1 )</td>
<td>1.28</td>
<td>1.30</td>
<td>-1.60</td>
</tr>
<tr>
<td><strong>Mean</strong> ( z_2 - z_1 )</td>
<td>1.30</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> ( z_2 - z_1 )</td>
<td>0.49</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> ( z_3 - z_2 )</td>
<td>1.22</td>
<td>1.19</td>
<td></td>
</tr>
<tr>
<td><strong>Mean</strong> ( z_3 - z_2 )</td>
<td>1.24</td>
<td>1.22</td>
<td>0.80</td>
</tr>
<tr>
<td><strong>StDev</strong> ( z_3 - z_2 )</td>
<td>0.52</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> ( z_3 - z_1 )</td>
<td>2.54</td>
<td>2.56</td>
<td>-0.38</td>
</tr>
<tr>
<td><strong>Mean</strong> ( z_3 - z_1 )</td>
<td>2.55</td>
<td>2.56</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> ( z_3 - z_1 )</td>
<td>0.52</td>
<td>0.52</td>
<td></td>
</tr>
</tbody>
</table>
All curves have $\log(\text{cond}) \in [15, 16]$;

- $z_j =$ imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>64 Rank 2 Curves</th>
<th>23 Rank 4 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median $z_2 - z_1$</td>
<td>1.26</td>
<td>1.27</td>
<td></td>
</tr>
<tr>
<td>Mean $z_2 - z_1$</td>
<td>1.36</td>
<td>1.29</td>
<td>0.59</td>
</tr>
<tr>
<td>StDev $z_2 - z_1$</td>
<td>0.50</td>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_2$</td>
<td>1.22</td>
<td>1.08</td>
<td>1.35</td>
</tr>
<tr>
<td>Mean $z_3 - z_2$</td>
<td>1.29</td>
<td>1.14</td>
<td></td>
</tr>
<tr>
<td>StDev $z_3 - z_2$</td>
<td>0.49</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>Median $z_3 - z_1$</td>
<td>2.66</td>
<td>2.46</td>
<td></td>
</tr>
<tr>
<td>Mean $z_3 - z_1$</td>
<td>2.65</td>
<td>2.43</td>
<td>2.05</td>
</tr>
<tr>
<td>StDev $z_3 - z_1$</td>
<td>0.44</td>
<td>0.42</td>
<td></td>
</tr>
</tbody>
</table>
All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of the } j^{th} \text{ norm zero above the central point}$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>701 Rank 2 Curves</th>
<th>64 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median</strong> $z_2 - z_1$</td>
<td>1.30</td>
<td>1.26</td>
<td>0.69</td>
</tr>
<tr>
<td><strong>Mean</strong> $z_2 - z_1$</td>
<td>1.34</td>
<td>1.36</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_2 - z_1$</td>
<td>0.51</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> $z_3 - z_2$</td>
<td>1.19</td>
<td>1.22</td>
<td>1.39</td>
</tr>
<tr>
<td><strong>Mean</strong> $z_3 - z_2$</td>
<td>1.22</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_3 - z_2$</td>
<td>0.47</td>
<td>0.49</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong> $z_3 - z_1$</td>
<td>2.56</td>
<td>2.66</td>
<td>1.93</td>
</tr>
<tr>
<td><strong>Mean</strong> $z_3 - z_1$</td>
<td>2.56</td>
<td>2.65</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong> $z_3 - z_1$</td>
<td>0.52</td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>
New Model for Finite Conductors

- Replace conductor $N$ with $N_{\text{effective}}$.
  - Arithmetic info, predict with $L$-function Ratios Conj.
  - Do the number theory computation.

- Excised Orthogonal Ensembles.
  - $L(1/2, E)$ discretized.
  - Study matrices in $\text{SO}(2N_{\text{eff}})$ with $|\Lambda_A(1)| \geq c e^N$.

- Painlevé VI differential equation solver.
  - Use explicit formulas for densities of Jacobi ensembles.
  - Key input: Selberg-Aomoto integral for initial conditions.
Modeling lowest zero of $L_{E11}(s, \chi_d)$ with $0 < d < 400,000$

Lowest zero for $L_{E11}(s, \chi_d)$ (bar chart), lowest eigenvalue of $\text{SO}(2N)$ with $N_{\text{eff}}$ (solid), standard $N_0$ (dashed).
Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$

Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of $\text{SO}(2N)$: $N_{\text{eff}} = 2$ (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.
Cuspidal Maass Forms
(Joint with Levent Alpoge)
**Maass Forms**

**Definition: Maass Forms**

A Maass form on a group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is a function $f: \mathcal{H} \rightarrow \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. $f$ vanishes at the cusps of $\Gamma$, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1 - s) > 0$, where

\[
\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$. 
Write Fourier expansion of Maass form $u_j$ as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi|n|y) e^{2\pi i nx}.$$

Define $L$-function attached to $u_j$ as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_j(p)}{p^s} \right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p)$, $\alpha_j(p)\beta_j(p) = 1$, $\lambda_j(1) = 1$.

Also,

$$\lambda_j(p) \ll p^{7/64}.$$
Kuznetsov Trace Formula

\[ \sum_j \frac{h(t_j)}{||u_j||^2} \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_\mathbb{R} \frac{h(r)}{\tau(m, r) \tau(n, r)} \frac{h(r)}{\cosh(\pi r)} \, dr = \]

\[ \frac{\delta_{n, m}}{\pi^2} \int_\mathbb{R} r \tanh(r) h(r) \, dr + \frac{2i}{\pi} \sum_{c \geq 1} S(n, m; c) \int_\mathbb{R} J_{ir} \left( \frac{4\pi \sqrt{mn}}{c} \right) \frac{h(r) r}{\cosh(\pi r)} \, dr \]

where

\[ \tau(m, r) = \pi^{1/2+ir} \Gamma(1/2 + ir)^{-1} \zeta(1 + 2ir)^{-1} n^{-1/2} \sum_{ab=|m|} (\frac{a}{b})^r. \]

\[ S(n, m; c) = \sum_{0 \leq x \leq c-1, \gcd(x, c)=1} e^{2\pi i (nx + mx^*) / c} \]

\[ J_{ir}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + ir + 1)} \left( \frac{1}{2} x \right)^{2m + ir}. \]
Theorem (Alpoge-Miller 2012)

For $h_T(r) = \frac{r}{T} h(ir / T) / \sinh(\pi r / T)$ the Katz-Sarnak conjecture holds for level 1 cuspidal Maass forms for test functions whose Fourier transform is supported in $(-4/3, 4/3)$.

- Write $\int_{-\infty}^{\infty} J_{2ir}(X) \frac{r h_T(r)}{\cosh(\pi r)} dr$ as sum of $J_{2k+1}(X)$ and $h_T$ at imaginary arguments and $J_{2kT}(X)$ and $h$ at $k$.

- Bound contributions from sums. Apply Poisson summation, analyze, and Poisson summation again.

- Key steps: Taylor expanding, Fourier transform identities relating differentiation and multiplication.
Cuspidal Newforms
(Joint with C. Hughes, G. Iyer, N. Triantafillou)
Modular Form Preliminaries

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \ c \equiv 0(N) \right\} \]

\( f \) is a weight \( k \) holomorphic cuspform of level \( N \) if

\[ \forall \gamma \in \Gamma_0(N), \ f(\gamma z) = (cz + d)^k f(z). \]

- Fourier Expansion: \( f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \)
  \( L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}. \)
- Petersson Norm: \( \langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^{k-2} \, dx \, dy. \)
- Normalized coefficients:
  \[ \psi_f(n) = \sqrt{\frac{\Gamma(k - 1)}{(4\pi n)^{k-1}}} \frac{1}{\|f\|} a_f(n). \]
Elliptic Curves

Maass Forms

Cuspidal Newforms

Recap/Refs

Excised Ensembles

Modular Form Preliminaries: Petersson Formula

\( B_k(N) \) an orthonormal basis for weight \( k \) level \( N \). Define

\[
\Delta_{k,N}(m, n) = \sum_{f \in B_k(N)} \psi_f(m) \overline{\psi}_f(n).
\]

Petersson Formula

\[
\Delta_{k,N}(m, n) = 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m, n, c)}{c} J_{k-1} \left( 4\pi \frac{\sqrt{mn}}{c} \right) + \delta(m, n).
\]
2-Level Density

\[ \int_{x_1=2}^{R^\sigma} \int_{x_2=2}^{R^\sigma} \phi \left( \frac{\log x_1}{\log R} \right) \phi \left( \frac{\log x_2}{\log R} \right) J_{k-1} \left( 4\pi \frac{\sqrt{m^2 x_1 x_2 N}}{c} \right) \frac{dx_1 dx_2}{\sqrt{x_1 x_2}} \]

Change of variables and Jacobian:

\[
\begin{align*}
U_2 &= x_1 x_2 \\
U_1 &= x_1 \\
X_2 &= \frac{u_2}{u_1} \\
X_1 &= U_1
\end{align*}
\]

\[
\left| \frac{\partial x}{\partial u} \right| = \begin{vmatrix} 1 & 0 & 1 \end{vmatrix} = \frac{1}{u_1}.
\]

Left with

\[ \int \int \hat{\phi} \left( \frac{\log u_1}{\log R} \right) \hat{\phi} \left( \frac{\log \left( \frac{u_2}{u_1} \right)}{\log R} \right) \frac{1}{\sqrt{u_2}} J_{k-1} \left( 4\pi \frac{\sqrt{m^2 u_2 N}}{c} \right) \frac{du_1 du_2}{u_1} \]
2-Level Density

Changing variables, $u_1$-integral is

$$\int_{\sigma}^\infty \hat{\phi}(w_1) \hat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$ 

Support conditions imply

$$\psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \hat{\phi}(w_1) \hat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$ 

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi \frac{\sqrt{m^2 u_2 N}}{c}\right) \psi_2\left(\frac{\log u_2}{\log R}\right) \frac{du_2}{\sqrt{u_2}}.$$
$n$-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m} \int_0^\infty J_{k-1}(x) \widehat{\Phi}_n \left( \frac{2 \log(bx\sqrt{N}/4\pi m)}{\log R} \right) \frac{dx}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an $n$-dimensional integral of $\phi_1(x_1) \cdots \phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.
Theorem (Iyer-Miller-Triantafillou):
The $n$-level densities agree for $\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:
Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let $\Psi$ be an even Schwartz function with $\text{supp}(\hat{\Psi}) \subset (-2, 2)$. Then

$$
\sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2, b)R(1, b)}{\varphi(b)} \int_{y=0}^\infty J_{k-1}(y) \hat{\Psi} \left( \frac{2 \log(by\sqrt{N}/4\pi m)}{\log R} \right) \frac{dy}{\log R}
$$

$$
= -\frac{1}{2} \left[ \int_{-\infty}^\infty \psi(x) \frac{\sin 2\pi x}{2\pi x} \, dx - \frac{1}{2} \psi(0) \right] + O \left( \frac{k \log \log kN}{\log kN} \right),
$$

where $R = k^2 N$, $\varphi$ is Euler’s totient function, and $R(n, q)$ is a Ramanujan sum.
Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

Number Theory Side: Iyer-Miller-Triantafillou:
$\text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$
New Results: Number Theory Side: Iyer-Miller-Triantafillou:

\( \text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right) \)

**Theorem**

*Fix* \( n \geq 4 \) *and let* \( \phi \) *be an even Schwartz function with* \( \text{supp}(\hat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right) \). *Then, the* \( n \)-th centered moment of the 1-level density for holomorphic cusp forms is*

\[
\frac{1 + (-1)^n}{2} (-1)^n (n - 1)!! \left( 2 \int_{-\infty}^{\infty} \hat{\phi}(y)^2 |y| dy \right)^{n/2} \\
\pm (-2)^{n-1} \left( \int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2} \hat{\phi}(0)^n \right) \\
\pm (-2)^{n-1} n \left( \int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1) \frac{\sin(2\pi x_1 (1 + |x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2} \phi^n(0) \right).
\]

Agrees with RMT.
**n-Level Density: Katz-Sarnak Determinant Expansions**

**Example:** \( \text{SO(even)} \)

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \det \left( K_1(x_j, x_k) \right)_{1 \leq j, k \leq n} \, dx_1 \cdots dx_n,
\]

where

\[
K_1(x, y) = \frac{\sin \left( \pi(x - y) \right)}{\pi(x - y)} + \frac{\sin \left( \pi(x + y) \right)}{\pi(x + y)}.
\]

**Problem:** \( n \)-dimensional integral - looks very different.
Easier to work with cumulants.

\[ \sum_{n=1}^{\infty} \frac{C_n (it)^n}{n!} = \log \hat{P}(t), \]

where \( P \) is the probability density function.

\[ \mu'_n = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu'_{n-m}, \]

where \( \mu'_n \) is uncentered moment.
Manipulating determinant expansions leads to analysis of

\[ K(y_1, \ldots, y_n) = \sum_{m=1}^{n} \sum_{\lambda_1+\ldots+\lambda_m=n}^{\lambda_j \geq 1} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \prod_{\ell=1}^{m} \chi\left\{|\sum_{j=1}^{n} \eta(\ell, j) \epsilon_j y_j| \leq 1\right\}, \]

where

\[ \eta(\ell, j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell} \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell} \lambda_k. \end{cases} \]
New Result: Iyer-Miller-Triantafillou: Large Support:

\[ \text{supp}(\hat{\phi}) \subseteq \left( -\frac{1}{n-2}, \frac{1}{n-2} \right) \]

Hughes-Miller solved for \( \text{supp}(\hat{\phi}) \subseteq \left( -\frac{1}{n-1}, \frac{1}{n-1} \right) \).

New Complications: If \( \text{supp}(\hat{\phi}) \subseteq \left( -\frac{1}{n-2}, \frac{1}{n-2} \right) \),

1. \( \eta(\ell, j)e_jy_j \) need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed \( m, \lambda_j, \epsilon_j \)).

Solution: Double count terms and subtract a correcting term \( \rho_j \).
New Result: Iyer-Miller-Triantafillou: Large Support:
\[ \text{supp}(\hat{\phi}) \subseteq \left( -\frac{1}{n-2}, \frac{1}{n-2} \right) \]

After Fourier transform identities:

\[
C_n^{SO}(\phi) = \frac{(-1)^{n-1}}{2} \left( \int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2} \phi(0)^n \right) \\
+ \frac{n(-1)^n}{2} \left( \int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1) \\
\quad \sin(2\pi x_1 (1 + |x_2|)) \frac{dx_1 dx_2}{2\pi x_1} - \frac{1}{2} \phi^n(0) \right).
\]

Agrees with number theory!
Conclusion and Bibliography
Recap

- Understand compound families in terms of simple ones.

- Choose combinatorics to simplify calculations.

- Extending support often related to deep arithmetic questions.
References


- **Lower order terms in the 1-level density for families of holomorphic cuspidal newforms**, Acta Arithmetica **137** (2009), 51–98.


- **The low-lying zeros of level 1 Maass forms** (with Levent Alpoge), preprint 2013.

- **Moment Formulas for Ensembles of Classical Compact Groups** (with Geoffrey Iyer and Nicholas Triantafillou), preprint 2013.
Excised Orthogonal Ensembles
Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \text{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k-\theta)})(1 - e^{i(-\theta_k-\theta)}),$$

with $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$ the eigenvalues of $A$.

Motivated by the arithmetical size constraint on the central values of the $L$-functions, consider Excised Orthogonal Ensemble $T_\chi: A \in \text{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\chi).$
One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \ldots \int P(\theta, \theta_2, \ldots, \theta_N) d\theta_2 \ldots d\theta_N,$$

where $P(\theta, \theta_2, \ldots, \theta_N)$ is the joint probability density function of eigenphases.
One-Level Densities

One-level density $R_{1}^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_{1}^{G(N)}(\theta) = N \int \cdots \int P(\theta, \theta_2, \ldots, \theta_N) d\theta_2 \cdots d\theta_N,$$

where $P(\theta, \theta_2, \ldots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised orthogonal ensemble:

$$R_{1}^{T_{X}}(\theta_1) := C_{X} \cdot N \int_{0}^{\pi} \cdots \int_{0}^{\pi} H(\log |\Lambda_{A}(1, N)| - X) \times$$

$$\times \prod_{j<k}(\cos \theta_j - \cos \theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

and $C_{X}$ is a normalization constant.
One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \cdots \int P(\theta, \theta_2, \ldots, \theta_N) d\theta_2 \cdots d\theta_N,$$

where $P(\theta, \theta_2, \ldots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised of the orthogonal ensemble is

$$R_1^{T\chi}(\theta_1) = \frac{C_\chi}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{N_r} \frac{\exp(-r \chi)}{r} R_1^{J_N}(\theta_1; r - 1/2, -1/2) dr,$$

where $C_\chi$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r - 1/2, -1/2) = N \int_0^\pi \cdots \int_0^\pi \prod_{j=1}^N w^{(r-1/2,-1/2)}(\cos \theta_j)$$

$$\times \prod_{j<k} (\cos \theta_j - \cos \theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble $J_N$ with weight function

$$w^{(\alpha,\beta)}(\cos \theta) = (1-\cos \theta)^{\alpha+1/2}(1+\cos \theta)^{\beta+1/2}, \quad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$
With $C_\mathcal{X}$ normalization constant and $P(N, r, \theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T,\mathcal{X}}(\theta) = \frac{C_\mathcal{X}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$

$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$

$$\times (1-\cos \theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N, r, \theta) \, dr.$$

Residue calculus implies $R_1^{T,\mathcal{X}}(\theta) = 0$ for $d(\theta, \mathcal{X}) < 0$ and

$$R_1^{T,\mathcal{X}}(\theta) = R_1^{SO(2N)}(\theta) + C_\mathcal{X} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \text{ for } d(\theta, \mathcal{X}) \geq 0,$$

where $d(\theta, \mathcal{X}) := (2N-1)\log 2 + \log(1-\cos \theta) - \mathcal{X}$ and $b_k$ are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, $\theta$ fixed, $R_1^{T,\mathcal{X}}(\theta) \to R_1^{SO(2N)}(\theta)$. 

57
Numerical check

**Figure:** One-level density of excized \( SO(2N) \), \( N = 2 \) with cut-off \( |\Lambda_A(1, N)| \geq 0.1 \). The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.
**Theory vs Experiment**

Figure: Cumulative probability density of the first eigenvalue from $3 \times 10^6$ numerically generated matrices $A \in SO(2N_{\text{std}})$ with $|\Lambda_A(1, N_{\text{std}})| \geq 2.188 \times \exp(-N_{\text{std}}/2)$ and $N_{\text{std}} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400,000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.