Determinantal Expansions in Random Matrix Theory and Number Theory

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http://web.williams.edu/Mathematics/sjmiller/public_html/jmm2013.html/

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Classical
Random Matrix Theory

Origins of Random Matrix Theory

Problem: What are energy levels of heavy nuclei?

Fundamental Equation: $H\psi_n = E_n\psi_n$.

Origins of Random Matrix Theory

Problem: What are energy levels of heavy nuclei?

Fundamental Equation: $H\psi_n = E_n\psi_n$.

Motivation: Statistical Mechanics - To compute quantity (e.g. pressure), calculate for each configuration, take the average.

Idea: Nuclear Physics - Choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

L-functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

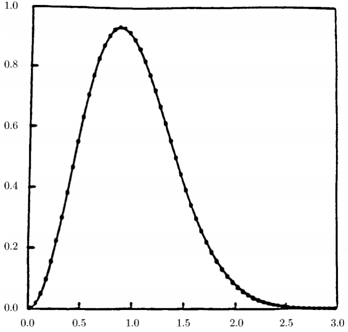
All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Why Study Zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$, the class number, i.e. the number of inequivalent binary quadratic forms with discriminant D, from L-functions with many central point zeros.

Random Matrix Theory - Number Theory Connection

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

Measures of Spacings: n-Level Density and Families

Let ϕ_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s, f)$ an L-function with zeros $\frac{1}{2} + i\gamma_f$ and conductor Q_f:

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\gamma_{j_1;f}\frac{\log Q_f}{2\pi}\right)\cdots\phi_n\left(\gamma_{j_n;f}\frac{\log Q_f}{2\pi}\right)$$

Measures of Spacings: *n*-Level Density and Families

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- Properties of n-level density:
  - ◇ Individual zeros contribute in limit
  - ◇ Most of contribution is from low zeros
  - ◇ Average over similar L-functions (family)

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, ϕ_k an even Schwartz function: $D_{n,\mathcal{F}}(\phi) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\gamma_{j_1;f}\frac{\log Q_f}{2\pi}\right)\cdots\phi_n\left(\gamma_{j_n;f}\frac{\log Q_f}{2\pi}\right)$$

As $N \to \infty$, n-level density converges to

$$\int \phi(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} = \int \widehat{\phi}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group (in the limit).

Correspondences

Similarities between *L*-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longleftrightarrow$	Proton
Support of test function	$\longleftrightarrow$	Proton Energy.

Number Theory: Extending the Support

Goal:

Prove n-level densities agree for $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:

Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\text{supp}(\widehat{\Psi}) \subset (-2, 2)$. Then

$$\sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2, b)R(1, b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x) \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right),$$

where $R = k^2N$, φ is Euler's totient function, and $R(n, q)$ is a Ramanujan sum.

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Typical Argument

If any prime is ‘special’, bound error terms

$$\ll \frac{1}{\sqrt{N}} \sum_{p_1,\dots,p_n \le N^\sigma} \sum_{m \le N^\epsilon} \sum_{r=1}^{\infty} \frac{m^2 r}{r p_1} \frac{\sqrt{p_1 \cdots p_n}}{r p_1 \sqrt{N}} \frac{1}{\sqrt{p_1 \cdots p_n}}$$

$$\ll N^{-1+\epsilon'} \left(\sum_{p \le N^\sigma} 1 \right)^{n-1} \ll N^{-1+(n-1)\sigma+\epsilon'}$$

Bounds fail for large support - new terms arise.

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Theorem

Fix $n \geq 4$ and let ϕ be an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$. Then, the nth centered moment of the 1-level density for even or odd holomorphic cusp forms is

$$\frac{1+(-1)^n}{2}(-1)^n(n-1)!!\left(2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy\right)^{n/2}$$
$$\pm(-2)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$
$$\mp(-2)^{n-1}n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2-\frac{1}{2}\phi^n(0)\right).$$

Agrees with Random Matrix Theory!

Pushing Beyond $2/n$.

Hypothesis T: There exist constants $A > 0$ and $\frac{1}{2} \leq \alpha < \frac{3}{4}$ such that

$$\sum_{\substack{p_1 p_2 \equiv a(c) \\ 1 \leq p_i \leq x_i}} e\left(\frac{2\sqrt{p_1 p_2}}{c}\right) \ll c^A (x_1 x_2)^\alpha.$$

Theorem

Assume Hypothesis T then, the 2nd centered moment of the 1-level density for all holomorphic cusp forms is $2\int_{-\infty}^{\infty} \hat{\phi}(y)^2 |y| dy$ *if* $\text{supp}(\hat{\phi}) \subset (-5/4, 5/4)$

Pushing Beyond $2/n$.

Theorem

Assume Hypothesis T then, the 2nd centered moment of the 1-level density for all holomorphic cusp forms is
$$2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy \text{ if } \operatorname{supp}(\hat{\phi}) \subset (-5/4, 5/4)$$

Theorem

The 2nd centered moment of the 1-level density for the full orthogonal ensemble is
$$2\int_{-\infty}^{\infty}\hat{\phi}(y)^2\min(1,|y|)dy + \left(\frac{1}{2}\int_{|y|\geq 1}\hat{\phi}(y)dy\right)^2$$

Disagreement! Hypothesis T is probably false.

Recap

Recap

1. Difficult to compare n-dimensional integral from RMT with NT in general. Harder combinatorics worthwhile to appeal to result from ILS.

2. Solve combinatorics by using cumulants; support restrictions translate to which terms can contribute.

3. Extend number theory results by bounding Bessel functions, Kloosterman sums, etc. New terms arise and match random matrix theory prediction.

4. Better bounds on percent of forms vanishing to large order at the center point.

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- The Québec-Maine Number Theory Conference organizers,
- Everyone in the audience.

Proofs

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

We want to evaluate

$$\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^{n}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$

$$=\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1}\left[\prod_{i=1}^{n}\sum_{n_i}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]$$

$$\times\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)^{-s}G_{k-1}(s)ds$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

$$\sum_{r=1}^{\infty} \hat{\phi}\left(\frac{\log r}{\log R}\right) \frac{\chi_0(r)\Lambda(r)}{r^{(1+s)/2}\log R} = \phi\left(\frac{1-s}{4\pi i}\log R\right) + \mathcal{E}(s),$$

where

$$\mathcal{E}(s) = -\frac{1}{2\pi i}\int_{\mathfrak{Re}(z)=c} \phi\left(\frac{(2z-1-s)\log R}{4\pi i}\right) \frac{L'}{L}(z,\chi_0)dz.$$

For convenience, rename expressions, $X = Y + Z$.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

By the binomial theorem,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Re(s)=1} X^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\
=&\frac{1}{2\pi i}\int_{\Re(s)=1} (Y+Z)^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\
=&\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2\pi i}\int_{\Re(s)=1} Y^{n-j}Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds.
\end{aligned}$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

$Z = \mathcal{E}(s)$ is easy to bound (shift coutours), get

$$\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n-j}Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

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For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

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For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

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For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

Unfortunately, $Z = \mathcal{E}(s)$ is hard to compute with.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Big Idea! Recall $X = Y + Z$, write $Z = X - Y$.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Big Idea! Recall $X = Y + Z$, write $Z = X - Y$.

$$\frac{n}{2\pi i}\int_{\Re(s)=1} Y^{n-1}Z\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$= \frac{n}{2\pi i}\int_{\Re(s)=1} XY^{n-1}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$- \frac{n}{2\pi i}\int_{\Re(s)=1} Y^{n}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds.$$

Hughes-Miller handle Y^n term, XY^{n-1} term is similar.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\operatorname{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

When the dust clears, we see

$$\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^{n}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$
$$=(1-n)\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$
$$+n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1-|x_2|))}{2\pi x_1}dx_1dx_2\right.$$
$$\left.-\frac{1}{2}\phi^n(0)\right)+o(1)$$

Random Matrix Theory: New Combinatorial Vantage

n-Level Density: Katz-Sarnak Determinant Expansions

Example: SO(even)

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_1) \cdots \widehat{\phi}(x_n) \det \Big(K_1(x_j, x_k) \Big)_{1 \le j,k \le n} dx_1 \cdots dx_n,$$

where

$$K_1(x, y) = \frac{\sin\big(\pi(x-y)\big)}{\pi(x-y)} + \frac{\sin\big(\pi(x+y)\big)}{\pi(x+y)}.$$

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where

$$K_1(x, y) = \frac{\sin\big(\pi(x-y)\big)}{\pi(x-y)} + \frac{\sin\big(\pi(x+y)\big)}{\pi(x+y)}.$$

Problem: n-dimensional integral - looks very different.

Preliminaries

Easier to work with cumulants.

$$\sum_{n=1}^{\infty} C_n \frac{(it)^n}{n!} = \log \widehat{P}(t),$$

where P is the probability density function.

$$\mu'_n = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu'_{n-m},$$

where μ'_n is uncentered moment.

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1, \dots, y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\dots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!}$$

$$\sum_{\epsilon_1,\dots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \leq 1\right\}},$$

where

$$\eta(\ell, j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell} \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell} \lambda_k. \end{cases}.$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Hughes-Miller solved for $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$.

New Complications: If $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-2}, \frac{1}{n-2})$,

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

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Solution: Double count terms and subtract a correcting term ρ_j.

Generating Function Identity - $\lambda_i \geq 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} = \frac{1}{1-(1-e^x)} = \sum_{m=0}^{\infty} (1-e^x)^m$$

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$$= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\lambda=1}^{\infty} \frac{1}{\lambda!} x^{\lambda} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m}{\lambda_1! \cdots \lambda_m!} \right) x^n$$

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$$= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\lambda=1}^{\infty} \frac{1}{\lambda!} x^{\lambda} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m}{\lambda_1! \cdots \lambda_m!} \right) x^n$$

$$\Rightarrow (-1)^n = \sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m n!}{\lambda_1! \cdots \lambda_m!}.$$

New Result: Dealing With 'ρ_j's

All $\lambda_i, \lambda_i' \geq 1$.

$$\rho_j = \sum_{m=1}^{n} \sum_{l=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

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$$= n \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\lambda'_1+\cdots+\lambda'_{m-1}=n-1} \frac{(-1)^m}{m} \frac{(n-1)!}{\lambda'_1!\cdots\lambda'_{m-1}!}$$

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$$\sum_{j=1}^{n} \rho_j = \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$= n \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} \frac{(-1)^m}{m} \frac{(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

$$= n \sum_{m=1}^{n} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} (-1) \frac{(-1)^{m-1}(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

New Result: Dealing With 'ρ_j's

All $\lambda_i, \lambda_i' \geq 1$.

$$\sum_{j=1}^{n} \rho_j = \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$= n \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} \frac{(-1)^m}{m} \frac{(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

$$= n \sum_{m=1}^{n} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} (-1) \frac{(-1)^{m-1}(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

$$= n(-1)^n.$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

After Fourier transform identities:

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After Fourier transform identities:

$$
\begin{aligned}
C_n^{\mathrm{SO}}(\phi) \; = & \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right) \\
& + \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1)\right. \\
& \left. \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi^n(0)\right).
\end{aligned}
$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\text{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

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$$\begin{aligned} C_n^{\text{SO}}(\phi) \ &= \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right) \\ &+ \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1)\right. \\ &\left. \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi^n(0)\right). \end{aligned}$$

Agrees with number theory!