Low lying zeroes of Maass form L-functions

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- First let me tell you what I mean by "*low-lying zeroes*".

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- Which matrix? Philosophy: if you've no clue, choose randomly (hence *random matrix theory*).

- Amazingly, this isn't baloney. (Snaith, as posted by Bober on MathOverflow.)

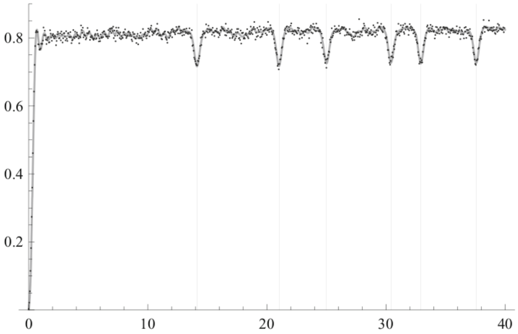

FIGURE 3. The dots represent the two-point correlation function of the raw Riemann zeros plotted using the first 100 000 zeros, computed by A. Odlyzko. The solid grey line is the prediction from the ratios conjecture, given in (4.19). The positions of the lowest six Riemann zeros are marked, coinciding with the dips in the two-point statistic.

- It turns out the zeroes of ζ do indeed act (statistically) like the eigenvalues of a random (large!) Hermitian matrix.

- No proof of RH, but opens up plenty of finer questions.

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- Yet more, $\pi_1^{\acute{e}t}$ also acts on these groups via monodromy (and this was essential in Deligne's work!). So, since you'd like to finish off your proof of RH in the next five minutes:

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- who are the corresponding groups in "real life"??

Definition (Katz-Sarnak)

Let $\phi : \mathbb{R} \to \mathbb{R}$ Schwartz with compactly supported Fourier transform. For f a cuspidal automorphic form, write

$$D_1(\phi, f) := \sum_{\rho} \phi\left(\frac{\gamma \log c_f}{2\pi}\right),$$

the *one-level density* of f, with ρ the nontrivial zeroes of $L(s, f)$, and c_f the analytic conductor of f.

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- (Philosophy: the more averaging, the better the shot at proving something.) For $\mathcal{F}$ a family of cuspidal automorphic forms, choose a weighting function h and write:

$$\mathcal{D}_1(\phi, h) := \operatorname*{Avg}_{\mathcal{F}}(D_1(\phi, f), h(f)),$$

the *averaged one-level density* of $\mathcal{F}$.

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  — that is, the n-level correlation functions of the three agree.

- But the one (indeed, n)-level densities of these ensembles *differ*. (So we break universality.)

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- (We've found our monodromy group!)

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- Main methods: trace formulae.

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- — and we have analogues of the trace formulae used by Iwaniec-Luo-Sarnak for these guys, too.

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$$\sum_u \frac{h(t_u)}{||u||^2}\lambda_m\bar{\lambda}_n = \frac{\delta_{m,n}}{\pi}\int_{\mathbb{R}} rh(r)\tanh(r)dr$$
$$-\frac{1}{\pi}\int_{\mathbb{R}} m^{ir}\sigma_{ir}(m)n^{-ir}\sigma_{-ir}(n)\frac{h(r)}{|\zeta(1+2ir)|^2}dr$$
$$+\frac{2i}{\pi}\sum_{c\geq 1}\frac{S(m,n;c)}{c}\int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{rh(r)}{\cosh(\pi r)}dr,$$

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$$\sum_u \frac{h(t_u)}{||u||^2} \lambda_m \bar{\lambda}_n = \frac{\delta_{m,n}}{\pi} \int_{\mathbb{R}} r h(r) \tanh(r) dr$$
$$- \frac{1}{\pi} \int_{\mathbb{R}} m^{ir} \sigma_{ir}(m) n^{-ir} \sigma_{-ir}(n) \frac{h(r)}{|\zeta(1 + 2ir)|^2} dr$$
$$+ \frac{2i}{\pi} \sum_{c \geq 1} \frac{S(m, n; c)}{c} \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{r h(r)}{\cosh(\pi r)} dr,$$

  the sum taken over the Hecke-Maass eigenforms of level 1.
- Here λ_m are the Hecke eigenvalues, $||u||$ is the L^2 norm of u (on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$), $\sigma_k(n) := \sum_{d|n} n^k$, $J_\alpha(x)$ is the usual Bessel function, and $S(m, n; c)$ is the usual Kloosterman sum.

- Handling the first two terms is entirely straightforward.

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- It is the last term ($\sum_{c\geq 1} \frac{S(m,n;c)}{c} \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{rh(r)}{\cosh(\pi r)} dr$) that has all the arithmetic.

- Now for the new stuff.

Theorem (A.-M.)

The Katz-Sarnak density conjecture holds for the family of Maass forms of level 1*, with symmetry type* O*. More precisely, it holds once* ϕ *is restricted to have Fourier transform supported in* $(-\frac{4}{3}, \frac{4}{3})$*.*

- Let me loosely sketch the argument.
- The problem immediately reduces to one of finding *lots* of cancellation in that last term.

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- First insight: choose the weight function carefully.
- Second insight: move the integral along $\mathbb{R}$ down to $\mathbb{R} - i\infty$ — i.e., replace it with a sum over residues.
- One then has to show (for h quite nice, $X := \frac{4\pi\sqrt{p}}{c}$):

$$\sum_p \frac{\log p}{\sqrt{p}} \hat{\phi}\left(\frac{\log p}{\log T}\right) \sum_{c \gg \frac{\sqrt{p}}{T}} \frac{S(p,1;c)}{c}$$
$$\cdot \left(\sum_{|\alpha| < \frac{T}{2}} \sum_{k \in \mathbb{Z}} e\left(\frac{k\alpha}{2T}\right) J_k(X) h\left(\frac{k}{2T}\right) \right)$$
$$\ll T^{1-\delta}$$

for some $\delta > 0$. (The conditions ensure $X \ll T$.)

- One also has to deal with $X \gg T$ (indeed, this is where we "break $(-1, 1)$") — but, once we get up to $X \asymp T$, moving slightly past this range is easier, essentially because there are quite nice asymptotic expansions for Bessel functions of purely imaginary order.

- Anyway, back to $X \ll T$. A key result of Iwaniec's (as used in Iwaniec-Luo-Sarnak's proof) handles a related sum with Poisson summation. (Point: Bessel functions of integer order are secretly Fourier coefficients.)

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- Following Iwaniec, one gets (for a different h):

$$\sum_{|\alpha|<\frac{T}{2}} \int_{-\infty}^{\infty} \hat{h}(t) e\left(Y \sin\left(\frac{\pi t}{T} + \frac{\pi \alpha}{T}\right)\right) dt.$$

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  and another similar term.
- Fifth insight: apply Poisson summation again!!! Then apply the method of stationary phase (in the spirit of van der Corput, but we are walking on too thin a rope to incur the error term of applying a canned theorem). This is enough for a good bound.

- Thanks so much for listening!!!
- Thanks also to the organizers of the Joint Mathematics Meetings, as well as the NSF, SMALL, and everyone else involved.