Most Sets are Balanced in Finite Groups

Kevin Vissuet (UCSD) and Steven J. Miller (Williams College)

http://web.williams.edu/Mathematics/sjmiller/public_html/jmm2013.html/

General Contributed Paper Session
Research in Number Theory
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Summary

- History
- Main Result and Proof
- Why the Dihedral Group is Special

Introduction

Statement

S finite set of integers, $|S|$ its size. Form

- Sumset: $S + S = \{a_i + a_j : a_j, a_j \in S\}$.
- Difference set: $S - S = \{a_i - a_j : a_j, a_j \in S\}$.

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Definition

We say S is difference dominated if $|S - S| > |S + S|$, balanced if $|S - S| = |S + S|$ and sum dominated (or an MSTD set) if $|S + S| > |S - S|$.

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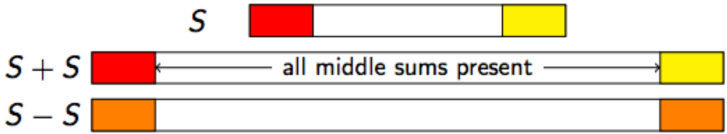
Sum Dominated sets are rare but do occur.
Conway: $\{0, 2, 3, 4, 7, 11, 12, 14\}$

Intuition

- Key Idea: In the $\mathbb{Z}$ case, fringe matters most, middle sums and differences are present with high probability.



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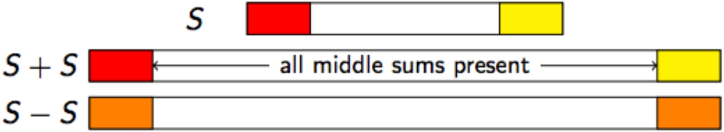

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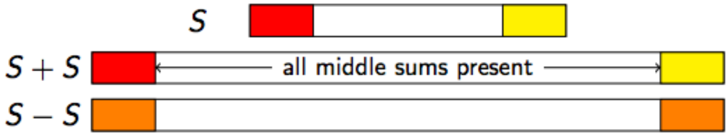

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- In a finite group there is no fringe. So the "largely irrelevant" is the only thing that can be relevant.

Main Result

Theorem

Let G be a group and let $S \subseteq G$. As $|G| \to \infty$
$\mathbb{P}(S + S = S - S = G) \to 1.$

Thus, as an immediate consequence, most set are balanced in finite groups.

Proof

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This is not entirely trivial to compute since there are some slight dependency issues for example when we have $xy = zx = g$.

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Claim: The number of subsets S of the "chain" elements $\{a_0, a_1, \ldots, a_n\}$ such that $g \notin S + S$ is the nth Lucas number.

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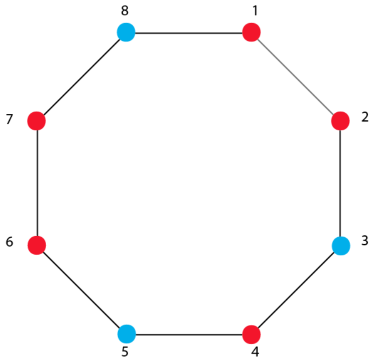
Claim: The number of subsets S of the "chain" elements $\{a_0, a_1, \ldots, a_n\}$ such that $g \notin S + S$ is the nth Lucas number.

To see this we look at a n-sided polygon.

The number of subsets such that $g \notin S + S$ is equal to the number of ways we can color the vertices of an n-polygon red or blue such that no two adjacent vertices are blue.

For example, here we have a possible coloring for a chain corresponding to $\bar{7} \in \mathbb{Z}/8\mathbb{Z}$.

Blue signifies that element is in S.



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We also know that the n^{th} Lucas Number is given by $L(n) = \phi^n + (-1/\phi)^n$ where ϕ is the golden ratio.

Thus, the n^{th} lucas number can be bounded above by $L(n) < 1.8^n$.

$$\mathbb{P}(g \notin S + S) = \frac{\prod L(n_i)}{2^{|G|}} \leq \frac{1.8^{\sum n_i}}{2^{|G|}} = \left(\frac{1.8}{2}\right)^{|G|}$$

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$$\begin{aligned}\mathbb{P}(S+S \neq G) &= \mathbb{P}(\cup_{g\in G} g \notin S+S) \\ &\leq \sum_{g\in G} \mathbb{P}(g \notin S+S) \\ &\leq |G|(1.8/2)^{|G|}\end{aligned}$$

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So as $|G| \to \infty$ we have that $\mathbb{P}(S+S \neq G) = 0$. $\square$

Why the Dihedral Group is Special

Recall that in the integer case there exists many more difference dominated sets than sum dominated sets.

This is no longer necessarily the case in finite groups.

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Conjecture: For any Dihedral Group, there exists more sum dominated subsets than difference dominated subsets.

Some Intuition on Why This Should Be True

We know that a presentation for the dihedral group is D_{2n} is $\langle a, b | a^n = abab = b^2 = e \rangle$.

The thing to notice is that at least half the elements in D_{2n} are of order 2.

So for many elements $x = x^{-1}$

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Note that the difference in what contributes to the sumset and diffset is $R - R$ which contributes to the diffset and $-R + F$ and $R + R$ which contribute to the sunset.

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Conclusion

A finite group acts different than the integers because a finite group does not have fringe elements.

However, if we let S be a random subset such that for each element in $g \in G$, $\mathbb{P}(g \in S) = 1/|G|$ then trivially $\mathbb{P}(S + S = G) < 1$.

So a question to ask is, with what constant probability does the phase transition occur.

Acknowledgements

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Bibliography

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