The M&M Game: A Study in Stochastic Processes and Probabilistic Models

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What is the M&M Game and why?

An interesting question

If two people are born on the same day, will they die on the same day?

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Model with the M&M Game.

- *Two people start with the same number of M&M's*
- *On every turn they each flip a fair coin simultaneously, eating an M&M if and only if a head is tossed.*
- *The players continue tossing coins until no one has any M&M's left, and the last player(s) to run out of M&M's wins.*

Previous Results

For each round, we have

- Player one eats an M&M.
- Player two eats an M&M.
- Each player eats an M&M.
- Neither player eats an M&M

By summing over all the possible number of rounds,

$$\begin{aligned}\mathbb{P}(\text{tie}) &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \\ &= \sum_{n=k}^{\infty} \binom{n-1}{k-1}^2 \left(\frac{1}{2}\right)^{2n}.\end{aligned}$$

Previous Results

And it can be further simplified,

$$\mathbb{P}(\text{tie}) = \sum_{n=0}^{k-1} \binom{2k-n-2}{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{2k-n-1}.$$

By summing over all the possible number of rounds that at least one player eats an M&M.

Our Extensions

In our extenstions, we studied the probability of a tie for the following two game settings.

- Flipping Multiple Coins with Different Values.
- Exponentially Changing Head Probabilities.

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An analogy: Super Mario

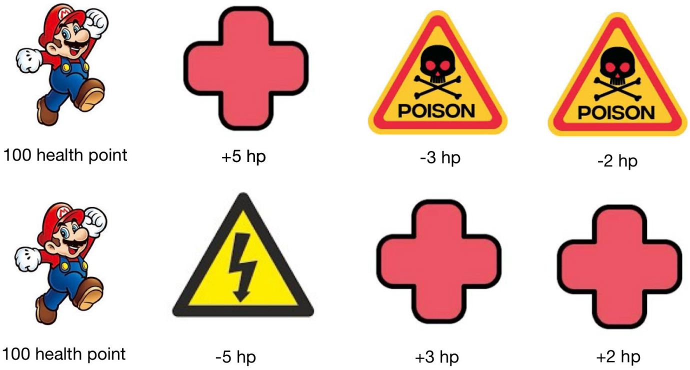

Figure: Do the two games have the same probability of running out of all their health points?

Result 1: Multiple Coins with Different Values

Suppose multiple coins are flipped in each round. Consider two games:

- **Game 1:** Coin values $5, -3, -2$, two players.
- **Game 2:** Coin values $-5, 3, 2$, two players.

Result 1: Multiple Coins with Different Values

Suppose multiple coins are flipped in each round. Consider two games:

- **Game 1:** Coin values $5, -3, -2$, two players.
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Define:

$$X_{i,t} = \text{Sum of values for coins tossed by player } t \text{ in round } i.$$

For Player 1 in Game 1:

$$\mathbb{P}(X_{i,1} = v, \text{ Game 1}) = \begin{cases} \frac{1}{8}, & v \in \{5, 3, 2, -2, -3, -5\} \\ \frac{1}{4}, & v = 0 \end{cases}$$

For player 1 in Game 2, same distribution:

$$\mathbb{P}(X_{i,1} = v, \text{ Game 2}) = \begin{cases} \frac{1}{8}, & v \in \{5, 3, 2, -2, -3, -5\} \\ \frac{1}{4}, & v = 0 \end{cases}$$

What the game looks like?

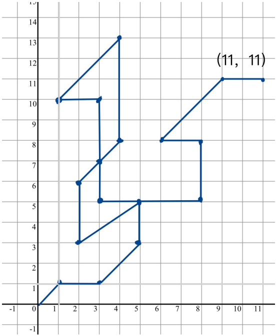

Figure: Two players starting with 11 M&M's, the trajectory keeps track of their M&M's at each round with player one's data in x-axis and player two's in y-axis.

Generalizing: Flipping Signs of Coin Values

General Setup

- **Game 1:** Coin values $a_1, a_2, \ldots, a_m$, two players.
- **Game 2:** Coin values $-a_1, -a_2, \ldots, -a_m$, two players.

Key Question: If $\sum_{i=1}^{m} a_i = 0$, do both games have the same probability of a tie?

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Key Question: If $\sum_{i=1}^{m} a_i = 0$, do both games have the same probability of a tie?

> Memoryless Property Failed: The case where both players get so lucky and always receive M&M''s cannot be simply eliminated. The game could last forever. $\Longrightarrow$ No closed formula for the probability of a tie.

Theorem

Let $(a_1, \ldots, a_m)$ represent an M&M Game with multiple fair coins such that $\sum_{i=1}^{m} a_i = 0$. Then, this game and its dual game, represented by $(-a_1, \ldots, -a_m)$, have the same probability of a tie.

Sketch of proof:

N_t : the number of turns for the t-th player in one game

$X_{i,t}$: Sum of values for coins tossed by player t in round i

$$\mathbb{P}(\text{Tie}) = \mathbb{P}(N_1 = N_2) = \sum_{n=1}^{\infty} \mathbb{P}(N_1 = n)\mathbb{P}(N_2 = n).$$

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When does $N_t = n$?
At the n-th round, the remaining M&M's of two players are all eliminated.
Also,

N_t **in the original game and its dual has the same distribution.**

Therefore, the same probability of ties for two games.□

Idea of the Proof

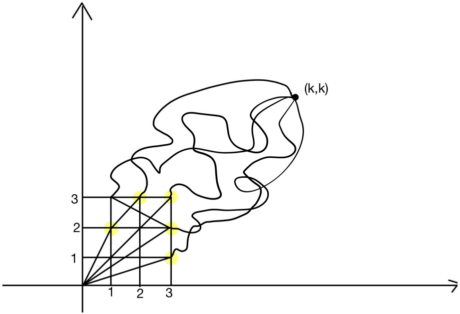

Figure: This graph shows the idea of the proof, each player starting with k M&M's, the set of coins are (3,-2,-1). Penultimate round (in yellow) hits the range of a box with side length 3 and both players eat all their remaining M&M's in the last round, there will be a tie.

*Note that there is an abuse of notation, the trajectories are not supposed to be smooth.

Simulation Result

Simulation:

Number of Ties Starting with 10, 15, 20 M&M's in 1 million runs			
Coin Values	Starting 10	Starting 15	Starting 20
(3,-2,-1), (-3,2,1)	87240 and 87117	87664 and 87601	88101 and 87845
$(5, 2, -3, -4)$, $(-5, -2, 3, 4)$	102352 and 102256	99467 and 98607	99739 and 100274

- The probability of tie actually doesn't depend on the initial number of M&M's. have the same probability.
- The numbers of ties in the original game and its dual game are indeed close.

Will the game run forever?

Memoryless Property Failed: The case where both players get so lucky and always receive M&M''s cannot be simply eliminated. The game could last forever. $\implies$ No closed formula for the probability of a tie.

But the game will end almost surely.

Gambler's ruin problem

Gambler's ruin problem: Suppose one player starts with k M&M's, he eats 1 M&M with probability $1/2$ and receives an M&M otherwise.
By symmetry,

$$\mathbb{P}(\text{Doubling his stack}) = \mathbb{P}(\text{Runs out all the M\&M's}) = \frac{1}{2}$$

Therefore,

$$\mathbb{P}(\text{His stack grows to } k \cdot 2^n) = (\frac{1}{2})^n.$$

While

$$\mathbb{P}(\text{Runs out all the M\&M's}) = 1 - (\frac{1}{2})^n \to 1 \text{ as } n \to \infty.$$

One player will run out of all his M&M's, and the game will end with a probability of 1.

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Motivation: Real-World Examples for Changing Probabilities

Why Study Exponentially Changing Probabilities?

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- In many real-world systems, the likelihood of an event increases as resources are consumed:
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- In the M&M game, the probability of flipping heads increases as the number of M&Ms decreases, modeling similar dynamic systems.

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Key Question

How does an exponentially increasing probability of flipping heads impact the likelihood of a tie?

Comparing Exponential and Gompertz Distributions

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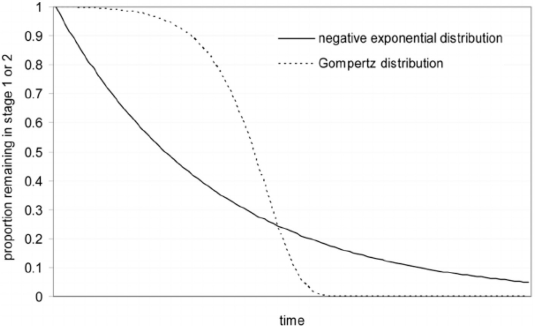

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- The probability of flipping heads is not constant; it evolves as M&Ms are consumed.
- Focus: How does the rate parameter λ affect the probability of a tie?

$$P(\text{heads}) = 1 - e^{-\lambda(n-m+1)}$$

- n: Initial number of M&Ms.
- m: Current number of M&Ms.
- λ: Rate parameter of the exponential distribution.

The probability of landing heads increases as the current M&M's decreases.

Player-Specific Probabilities: Different Numbers of M&Ms

Insight: Flipping Probability Depends on Remaining M&Ms

- For Player 1 with m_1 M&Ms remaining:

$$P_1(\text{heads}) = 1 - e^{-\lambda(n-m_1+1)}.$$

- For Player 2 with m_2 M&Ms remaining:

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Key Observation:

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Example

If Player 1 has 5 M&Ms left and Player 2 has 3 M&Ms left, the probability of flipping heads is higher for Player 2 since fewer M&Ms remain.

Simulation Results

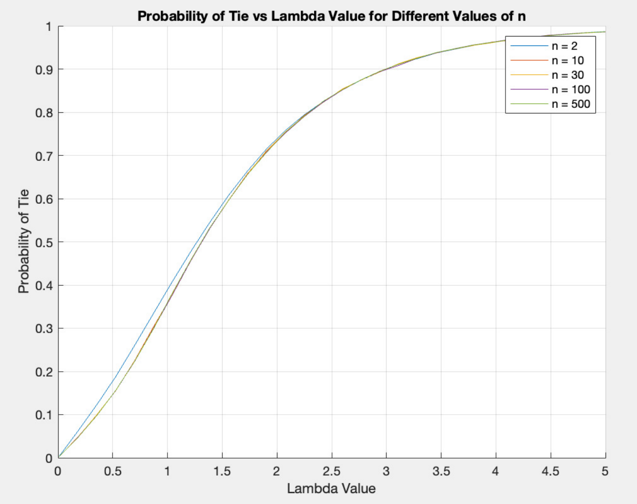

Figure: Using data from 2 million simulations, the following curves illustrate the fitted results for different values of λ

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- In this context, the tie probability $P_{\text{tie}}(\lambda)$ is modeled as:

$$P_{\text{tie}}(\lambda) = L \cdot \exp\left(-k \cdot \exp(-\lambda_0 \cdot \lambda)\right)$$

  where:
  - L: Maximum tie probability (asymptote).
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  - λ_0: Scale parameter controlling sensitivity to λ.
- Empirical fit: $L \approx 1, k \approx 1.22, \lambda_0 \approx 1$.

Gompertz Fit: Theoretical Insight

Theoretical Justification:

- The Gompertz form often arises in systems with:
  - Exponentially increasing failure rates (e.g., resource depletion).
- The M&M game exhibits similar dynamics as M&Ms decrease.

Conclusion

The empirical Gompertz fit aligns with known theoretical behaviors of resource depletion, hinting at a deeper justification.

Behavior of λ: Two Extreme Cases

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- **Case 1: λ is large (e.g., $\lambda \gg 1$)**
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Key Insight

λ controls how quickly the game progresses and determines the likelihood of a tie.

Intuitive Story: Loaded Coin vs Rigged Coin

The Story of the Game:

- λ **Large**: Imagine flipping a **loaded coin** that almost always lands heads.
- $\lambda = 0$: Imagine flipping a **rigged coin** that rarely lands heads.

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Takeaway

The behavior of the game changes dramatically with λ:

- λ Large: Fast game, likely tie.
- $\lambda = 0$: Slow game, unlikely tie.

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Summary

The M&M Game can be used to study two independent processes starting and ending simultaneously. For example,

- Two people born and die on the same day.
- Two gamblers starting with the same amount of fortune and go broke at the same time.
- Two chemical reactions in two separate containers starting and ending at the same time.
- One can think of more...

Summary of Extension 1

Theorem

Let $(a_1, \ldots, a_m)$ represent an M&M Game with multiple fair coins such that $\sum_{i=1}^{m} a_i = 0$. Then, this game and its dual game, represented by $(-a_1, \ldots, -a_m)$, have the same probability of a tie.

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- Memoryless property failed $\rightarrow$ No closed formula for the probability of a tie.
- Proof idea: Consider the penultimate round and same transition probability for each round.
- Gambler's ruin $\rightarrow$ The game will not last forever with almost surely.

Summary of Extension 2

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Key Takeaways:

- The tie probability evolves dynamically with λ:
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Impact of Initial M&M Count (n):

- For large n, the tie probability becomes almost independent of n, highlighting that λ is the dominant factor.
- Small n can have a noticeable effect, but this diminishes as $n \to \infty$, where the probability stabilizes.

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- Small n can have a noticeable effect, but this diminishes as $n \to \infty$, where the probability stabilizes.

Conclusion

The probability of a tie is primarily governed by λ, not the initial count of M&Ms (n), unless n is very small. This underscores the importance of dynamic probability models in analyzing real-world processes.

Future Work

- Study the decay pattern of the probability of a tie as the starting number of M&M's increases.

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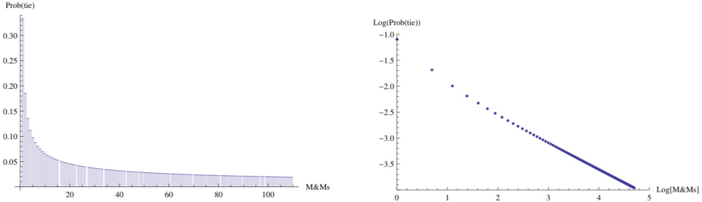

Figure: Probability of a tie for varying values of starting M&M's.

Authors in [BHM+17] constructed the line by looking at $50 \leq k \leq 110$, and found tremendous accuracy (the difference between the estimation and the probability is about .0002 when k equals 100,000!):

$$\log(\mathbb{P}(\text{Tie})) = -1.42022 - 0.545568 \log(k) \quad \text{or} \quad \mathbb{P}(\text{Tie}) \approx \frac{0.2412}{k^{0.5456}} \tag{1}$$

Future Work

- Varying probability of coins landing heads aside from exponential distribution.
- Consider the game with multiple coins, what if the coin values do not sum up to zero? How will the sum of coin values affect the probability of a tie?
- ...

Thank you

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Thanks for listening to our presentation.

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