Statistics of L-function zeros and exponential sums over primes

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Elementary, Watson!

"How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?"

— Sherlock Holmes

Zeros and Random Matrices

In recent decades a surprising connection has developed between zeros of L-functions and the eigenvalues of random matrices.

The **Katz-Sarnak** density conjectures formalize this correspondence and give insight into the statistics of zeros of families of L-functions.

Iwaniec, Luo, and Sarnak (ILS) studied a statistic for zeros called 1-level density, defined by

$$D(f;\phi) = \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi}\log c_f\right),$$

where ϕ is an even Schwartz function whose Fourier transform $\hat{\phi}$ has compact support.

One family of L-functions ILS looked at came from $H_k^*(N)$, the cuspidal newforms of weight k and level N. If $f \in H_k^*(N)$, then $f(z) = \sum_{n=1}^{\infty} \lambda_f(n)e(nz)$ and

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

Hypothesis S

ILS need an arithmetic hypothesis in order to extend the support of their test function beyond $(-1,1)$.

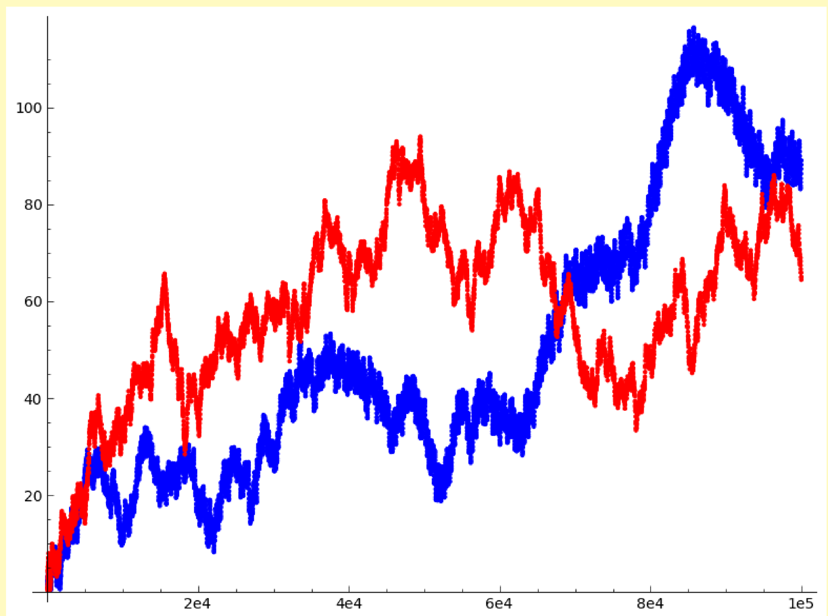
Conjecture 1 (Hypothesis S). *Let c be a positive integer, and let a be coprime to c. Then*

$$\sum_{\substack{p\le X\\ p\equiv a\ (c)}} e\left(\frac{2\sqrt{p}}{c}\right) \ll c^A X^\alpha$$

for some $A>0$ and $\alpha \in [\frac{1}{2},\frac{3}{4})$.

(Note: $A=0$ and $\alpha = 7/8$ was proved by Vinogradov.)

The following graph shows the Hypothesis S sum (in blue) and a sum over random real numbers (in red). Both plots show similar behavior.



We introduce $\log p$ weights to the sum. After summation by parts, we can use the L-function analogue of the so-called "explicit" formula of Riemann and von Mangoldt.

Hypothesis S (cont.)

We get a sum over oscillatory integrals.

$$\sum_{|\gamma|<T} \frac{1}{\rho}\int_2^X e^{i(4\pi\sqrt{x}+\gamma\log x)}\,dx$$

For $\gamma > 0$, we can use the standard technique of stationary phase. For $\gamma < 0$, the behavior is much harder to investigate. We need to use equidistribution results for $\gamma\log\gamma$.

2nd moment statistics

Last year, Triantafillou showed agreement between RMT and number theory for the second centered moment for orthogonal families of L-functions for test functions with $\hat{\phi}$ supported in $(-1,1)$.

Two kinds of terms appear in the calculations: $\Delta_k(1,1)$ and $\Delta_k(p_1p_2,1)$, where

$$\Delta_k(m,n) = \frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{\lambda_f(m)\lambda_f(n)}{L(1,\mathrm{sym}^2(f))}$$

Triantafillou showed that for $\hat{\phi}$ supported in $(-1,1)$, $\Delta_k(p_1p_2,1)$ does not contribute. To extend the support beyond $(-1,1)$, we need this term to contribute

$$4\int_1^\infty (1-y)\hat{\phi}(y)^2dy + \left(\int_1^\infty \hat{\phi}(y)dy\right)^2.$$

We conjecture that when p_1,p_2 are "close" (i.e., on a narrow band centered at the diagonal $p_1=p_2$), then $\Delta_k(p_1p_2,1)$ contributes the term on the left, and when p_1,p_2 are "far," then we get the term on the right.

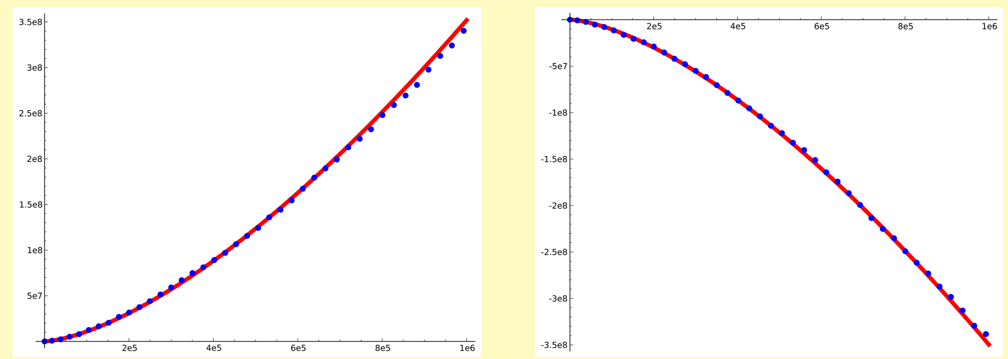
2nd moment statistics (cont.)

For X_1, X_2 far apart, we need to investigate a 2D analogue of the Hypothesis S sum:

$$\sum_{\substack{p_1\le X_1, p_2\le X_2\\ p_1p_2\equiv a\ (c)}} e\left(\frac{2\sqrt{p_1p_2}}{c}\right)\log p_1\log p_2.$$

In order to get agreement, we need a main term from the two dimensional sum. While we expect typical exponential sums to have square-root cancellation, there are lower order terms here that reinforce.

Let $X_1 = X_2 =: X$. Here are plots of the real (left) and imaginary (right) parts of the exponential sum for $c=1$ as we vary X.



Based on the plots, we expect the main term to be of the form $k\cdot X_1^{3/4}\cdot X_2^{3/4}$. This would bring us closer to agreement with RMT.

Finally, when p_1,p_2 are close, $\Delta_k(p_1p_2,1)$ should contribute the one-dimensional integral. Considering only $p_1=p_2$ is not enough; this gives a lower order term. As a result, we need p_1,p_2 on a narrow band, and we need the sum to stay roughly the same at small deviations, of square root order, from the diagonal. It seems improbable, but *Holmesian deduction* indicates the necessary veracity of this audacious claim!

Acknowledgments

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