Continued Fractions and Maclaurin's Inequalities

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1. Background

Definition 1.1. *Let $X = (x_1, x_2, \dots)$ be an arbitrary tuple of positive real numbers. Then for each $k \in \mathbb{N}$, the k^{th} elementary symmetric mean of the first n entries of X is defined to be*

$$S(X,n,k)^{1/k} := \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}}{\binom{n}{k}} \right)^{1/k}. \quad (1)$$

Note: $S(X,n,1)$ and $S(X,n,n)^{1/n}$ are the familiar arithmetic and geometric means, respectively.

Maclaurin's Inequalities For any tuple of positive numbers X, the following chain of inequalities holds:

$$S(X,n,1)^{1/1} \ge S(X,n,2)^{1/2} \ge \cdots \ge S(X,n,n)^{1/n} \quad (2)$$

Continued Fractions: Let α be any irrational number in $(0,1)$. Then we can write α uniquely as a *continued fraction*

$$\alpha = \cfrac{1}{a_1(\alpha) + \cfrac{1}{a_2(\alpha) + \cfrac{1}{\ddots}}} \quad (3)$$

where the $a_i(\alpha) \in \mathbb{N}^+$ are called the continued fraction digits of α.

Definition 1.2. *When $X = (a_1(\alpha), a_2(\alpha), \dots)$ is the sequence of continued fraction digits for α, we write $S(\alpha,n,k)$ instead of $S(X,n,k)$.*

Khinchin's Theorem (1933):
For almost every $\alpha \in (0,1)$,

$$\lim_{n\to\infty} S(\alpha,n,n)^{1/n} = 2.6854520\ldots =: K, \quad (4)$$

while

$$\lim_{n\to\infty} S(\alpha,n,1) = \infty. \quad (5)$$

The constant K is known as Khinchin's constant.

Khinchin's theorem, when combined with Maclaurin's Inequalities, opens up the possibility for a phase transition. As the left-most mean is almost always divergent, while the rightmost mean is almost always converging to the same number, we can expect some interesting changes in behavior as one takes more steps from the geometric mean toward the arithmetic mean.

Abstract

In this study, we analyze what happens to the means of typical continued fraction digits in the limit as one moves $f(n)$ steps away from either extreme. We show that the phase transition occurs when $f(n) = \Theta(n)$. That is, when $f(n) = o(n)$, for almost all α, $S(\alpha,n,n-f(n))^{\frac{1}{n-f(n)}}$ tends to Khinchin's constant K in the limit as $n \to \infty$, while $S(\alpha,n,f(n))^{\frac{1}{f(n)}}$ diverges. We also prove that for almost all α, $S(\alpha,n,cn)^{\frac{1}{cn}}$ is bounded in the limit. We prove that if the limit exists, it is a non-constant continuous function of c which satisfies a log-concavity-like condition.

2. Finding the Phase Transition

2.1 Outside of the critical region $f(n) = \Theta(n)$

We make use of the following result on symmetric means.

Theorem (Niculescu, 2001): If X is any tuple of positive real numbers, then for any $0 < t <$ and any $j,k \in \mathbb{N}$ such that $tj + (1-t)k \in \{1,\dots,n\}$, we have

$$S(X,n,tj+(1-t)k) \ge S(X,n,j)^t \cdot S(X,n,k)^{1-t}. \quad (6)$$

From (6) and a well-known result [2] about the rate of divergence of the arithmetic means of typical continued fraction digits, we can easily obtain the following result:

Theorem 1: For any arithmetic function $f(n)$ which is $o(\log\log n)$ and for almost all α, we have

$$\lim_{n\to\infty} S(\alpha,n,f(n))^{1/f(n)} = \infty. \quad (7)$$

We need another fact from Khinchin:

Theorem (Khinchin): For each $p < 0$, and almost all α,

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i(\alpha)^p \right)^{1/p} = K_p \quad (8)$$

where K_p is some constant in $(0,1)$.

Using this fact, we have shown with elementary arguments that

Theorem 2: For almost all α, and any $c \in (0,1]$, we have

$$K \le \limsup_{n\to\infty} S(\alpha,n,cn)^{1/cn} \le K^{1/c}(K_{-1})^{1-\frac{1}{c}}. \quad (9)$$

While this theorem does not give us much explicit information about the behavior of $S(\alpha,n,cn)^{1/cn}$, it does solve the case of $S(\alpha,n,n-f(n))^{\frac{1}{n-f(n)}}$, when $f(n) = o(n)$.

Corollary 2: If $f(n) = o(n)$, then for almost all α

$$\lim_{n\to\infty} S(\alpha,n,n-f(n))^{1/(n-f(n))} = K. \quad (10)$$

2.2 Inside the critical region $f(n) = \Theta(n)$

Definition 2.1. *For a fixed $\alpha \in (0,1)$, define the functions*

$$F_+^\alpha(c) = F_+(c) := \limsup_{n\to\infty} S(\alpha,n,cn)^{1/cn} \quad (11)$$

$$F_-^\alpha(c) = F_-(c) := \liminf_{n\to\infty} S(\alpha,n,cn)^{1/cn}. \quad (12)$$

Hypothesis 1: For almost all α, and each $c \in (0,1]$ we have

$$F_+(c) = F_-(c) = F(c) = \lim_{n\to\infty} S(\alpha,n,cn)^{1/cn}. \quad (13)$$

If we assume the rather weak Hypothesis 1, we can infer some nice properties of the function F.

Theorem 3: Assuming Hypothesis 1, the function $F : (0,1] \to [K,\infty)$ is continuous, monotone decreasing, and satisfies the following inequality for any $x,y,t \in (0,1]$:

$$\log F(tx + (1-t)y) \ge \frac{tx \log F(x) + (1-t)y \log F(y)}{tx + (1-t)y} \quad (14)$$

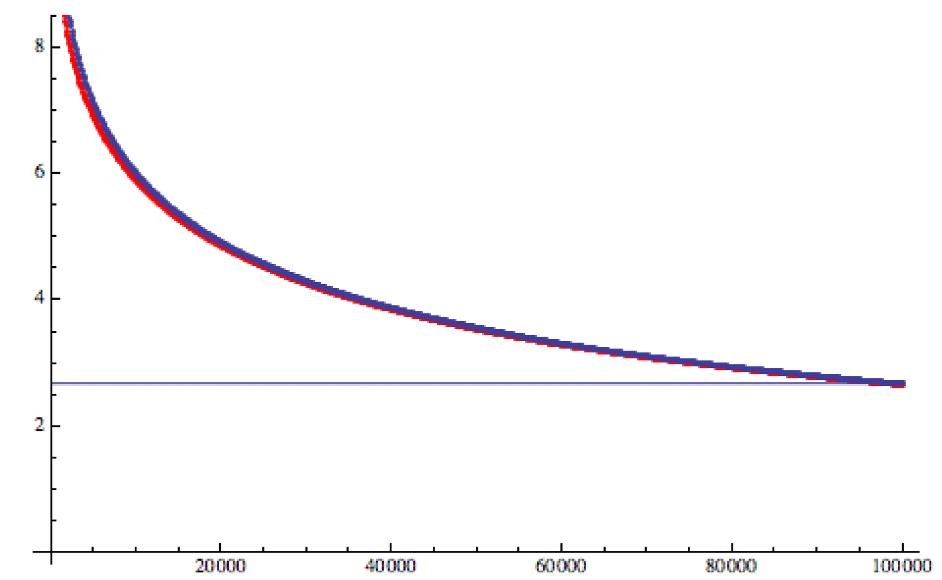
Note that the upper bound for $F(c)$ given by Theorem 2 satisfies (16), even when the inequality is replaced with equality.

3. Evidence for Hypothesis 1

Perhaps the strongest evidence for Hypothesis 1 comes from numerical simulations. Using the following recursion for the elementary symmetric polynomials $E[n,k]$:

$$E[n,k](x_1,\dots,x_n) = x_1 E[n-1,k-1](x_2,\dots,x_n) + E[n-1,k](x_2,\dots,x_n) \quad (15)$$

we were able to compute every elementary symmetric mean of the first 10^5 digits of numbers such as π and γ which are strongly believed to obey Khinchin's theorem.



Here, the red curve plots $S(\pi,10^5,k)^{1/k}$, while the blue curve is $S(\pi,10^4,k/10)^{10/k}$ as k varies from 1 to 10^5. The closeness of the two curves indicates that, for most values of $c = k/n > 0$, the means have settled close to their limiting values by the time $n \approx 10^4$. The horizontal line is Khinchin's constant K.

We also have bounds on how much $S(\alpha,n,cn)^{1/cn}$ can be influenced by a single term:

Theorem 4: For any $c \in (0,1]$, and almost all α, the difference between the n^{th} and the $n+1^{\text{st}}$ terms in the sequence $\{S(\alpha,n,cn)^{1/cn}\}_{n\in\mathbb{N}}$ is $O\left(\frac{\log n}{n}\right)$.

Another approach to understanding the function $F(c)$ is to examine the elementary symmetric means of periodic sequences.

Theorem 5: Let $X = (x_1,\dots,x_L,x_1,\dots)$ be a periodic sequence of positive real numbers with finite period L. Then for any $c \in (0,1]$, the limit

$$F_X(c) := \lim_{n\to\infty} S(X,n,cn)^{1/cn} \quad (16)$$

exists, and is a continuous function of c.

Combining this with another theorem from Khinchin:

Theorem (Khinchin): Let $k \in \mathbb{N}$. Then for almost all α,

$$\lim_{n\to\infty} \frac{|\{j \le n : a_j(\alpha) = k\}|}{n} = \log_2\left(1 + \frac{1}{k(k+2)}\right) \quad (17)$$

motivates the following definition:

Definition 3.1. *For each integer $d > 1$, we define a periodic sequence X_d via the following construction: for each $k \in \{2,3,4\dots,d\}$, let $\lfloor P(k) \cdot 10d^2 \rfloor$ of the first $10d^2$ digits of X_d equal k, and set the remaining of the first $10d^2$ equal to 1. Then make X_d periodic with period $10d^2$.*

Theorem 6: For any $d > 1$, $c \in (0,1]$, and almost all α,

$$F_{X_d}(c) \le \limsup_{n\to\infty} S(\alpha,n,cn)^{1/cn} \quad (18)$$

Since it is easy to get precise numerical estimates of $F_{X_d}(c)$, this gives us a way of obtaining information about $F_+^\alpha(c)$. In particular, we can show that, for c sufficiently small and d sufficiently large, $F_{X_d}(c)$ can be made arbitrarily large. This implies

Theorem 7: For any arithmetic function $f(n)$ which is $o(n)$, and almost all α, we have

$$\limsup_{n\to\infty} S(\alpha,n,f(n))^{1/f(n)} = \infty \quad (19)$$

If we assume Hypothesis 1, we can replace the limsup with a limit.

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