1.1 Introduction

MSTD: A set \( A \subset \mathbb{N} \) is called a more sums than differences set or MSTD if \( |A + A| > |A - A| \).

Martin and O’Bryan (2006): For \( n > 15 \), the proportion of subsets \( A \subset \{0, \ldots, n\} \) with \( |A + A| > |A - A| \) is at least \( 4 \cdot 10^{-7} \).

Example: Conway discovered the small MSTD set \( \{0,1,2,3,4,7,11,12,14\} \). Nathanson proved that this is in fact the smallest MSTD subset of \( \mathbb{N} \).

Explicit Constructions: Despite the fact that a uniformly random subset of \( \{0, \ldots, n\} \) is MSTD with positive probability, no one knows of an explicit family of MSTDs which has a non-trivial density. Miller, Orosz and Sheinerman have constructed an explicit family of subsets of \( \{0, \ldots, n\} \) of density \( \Omega(\log n/n) \). Zhai later used the combinatorial idea of bidirectional ballot sequences to improve this density to \( \Omega(n^{1/2}) \).

We can extend some of these results to subsets of higher dimensional lattices.

1.2 Our Results

Theorem (Probability in \( \mathbb{Z}^d \)): Let \( A \) be a uniformly random subset of the \( d \)-cube \( \{0, \ldots, n\}^d \). Then with positive probability, \( |A + A| > |A - A| \).

Proof idea: The probability that a given lattice point is in the sum or difference set approaches 1 as the point moves away from the corners, since points away from corners have many possible representations as sums and differences. Thus, if we demand that the corners of our sets have a given fringe profile which is MSTD, then with high probability, all middle sums and differences will be present, and so the sumset will win. By picking the fringes large enough, we can make sure the probability that \( A \) or \( A \) contains at least middle sums is at least some strictly positive constant independent of \( n \).

Note: This technique, which is a generalization of Martin and O’Bryan’s technique for one dimensional MSTD sets, also generalizes to higher dimensions and other shapes to give the following theorem:

Theorem (Arbitrary lattices): Let \( A \) be a uniformly random subset of some Cartesian product of intervals of \( d \)-dimensions. Then with positive probability, \( |A + A| > |A - A| \).

We also generalize the results of Zhao, who showed that in 1 dimension, the proportion of MSTD subsets of \( \{0, \ldots, n\} \) converges to a limit as \( n \to \infty \).

Theorem (Limiting proportions exist): The proportion of MSTD subsets of the \( d \)-cube \( \{0, \ldots, n\}^d \) converges to a positive limit.

From Monte Carlo experiments, the limiting proportion when \( d = 1 \) appears to be roughly \( 4.5 \cdot 10^{-1} \), while when \( d = 2 \), it appears to be more like \( 5 \cdot 10^{-1} \). This and other evidence leads us to conjecture...

Conjecture: The limiting proportion of subsets of the \( d \)-cube which are MSTD is monotonically decreasing in \( d \).

We have also explicitly constructed a family of MSTD subsets of the square with polynomial density:

2.2 Results

Theorem (Positive Limiting Probabilities Exist): For any \( p, p' \in [0, 1) \) such that \( p < p' \), the probability that \( A \equiv \{0, 1\} \times \{0, \ldots, n\} \) is MSTD approaches a positive limit \( P(n) \) as \( n \to \infty \).

Proof idea: We follow the main ideas of Zhao, who proved the result in the \( \{0, 1\} \)-case. We show that in the limit, an MSTD pair is rich (meaning the sumset and difference set contain all middle sums and differences) with probability 1. Rich MSTDs are easy to count by their minimal fringe profiles, which we use to show the limit exists.

Conjecture: If we constrain \( p_d = 1 - p_{d-1} \), then for any fixed \( p_1 \), the function \( P(n) \) is monotonically increasing and strictly increasing in \( p_d \).

2.3 Probability Decaying in \( N \)

Hegarty and Miller, 2008: When elements chosen with probability \( p(N) \to 0 \) as \( N \to \infty \), \( |A + A| \to |A - A| \) almost surely.

Our goal: Extend these results to correlated random subsets.

Lemma: The probability for the event that \( A \) or \( B \) are MSTDs in \( \mathbb{N} \) goes to 0 as \( N \to \infty \), \( |A + A| \to |A - A| \) almost surely.

3. References


E. Fein and S. Schneider, The Rules for Online Dating: Capturing the Heart of Mr. Right in Cyberspace, page 23.

P. V. Hegarty and S. J. Miller, When almost all sets are difference dominated, Random Structures and Algorithms 35 (2009), no. 1, 118–136.


