More Sums Than Differences Sets

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Introduction

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Given $A \subset \mathbb{Z}$, let

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\},$$
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Such sets are called *More Sums Than Differences (MSTD) sets*, or *sum-dominant sets*.

Fringe Elements

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Example

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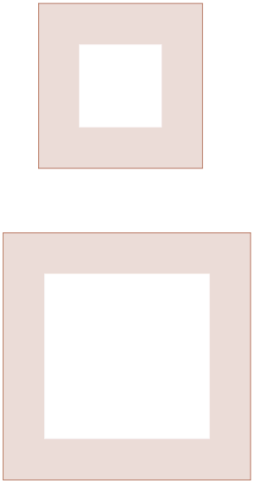
Martin and O'Bryant: for some fixed k, carefully choose fringe of A so that

$$\begin{aligned} &|\,(A + A) \cap ([0, k] \cup [2n - k, 2n])\,| \\ > &|\,(A - A) \cap ([-n, -n + k] \cup [n - k, n])\,| \end{aligned}$$

With positive probability, $[k + 1, 2n - k - 1] \subset A + A$ and A is MSTD.

Fringe in Higher Dimensions



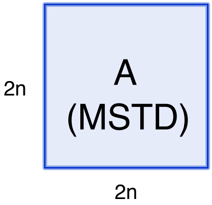
Positive Percentage in d Dimensions

Theorem

For any integer $d > 0$, there exists some constant $c_d > 0$ such that, for n large, the proportion of MSTD subsets A of $\{0, \ldots, n\}^d$ is greater than c_d.

Proof of positive proportion of MSTD sets is probabilistic.

Construction of Infinite Families of MSTD Sets



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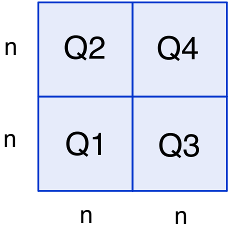

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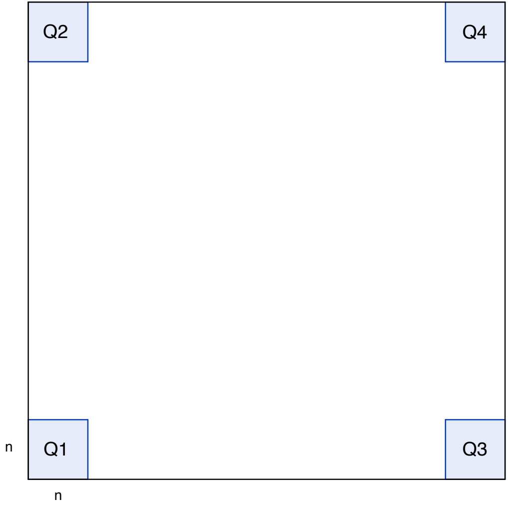

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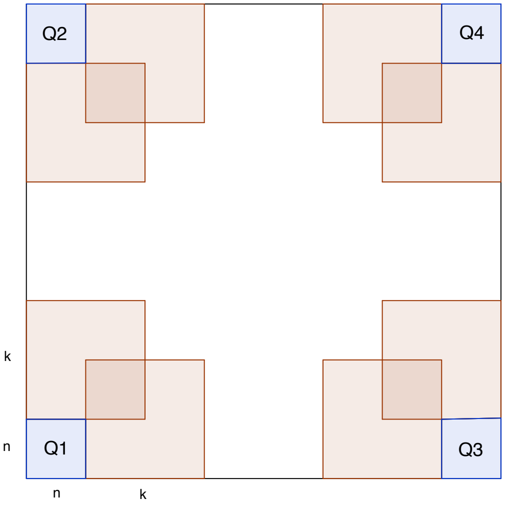

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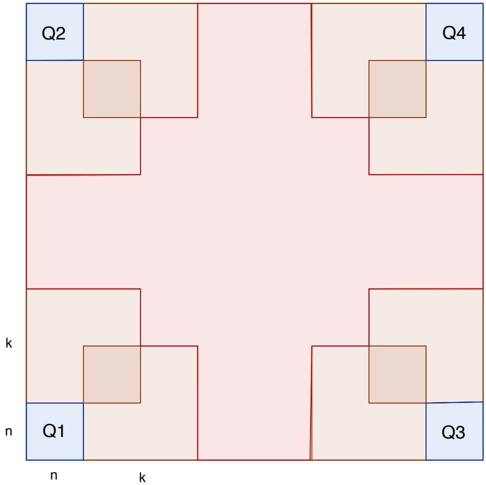

Correlated Random Pairs

All of the literature to date has looked at sums and differences of a set *with itself*. We investigate sums and differences of *pairs* of subsets $(A, B) \subset \{0, \dots, n\}$. We select such pairs according to the dependent random process:

$$P(a \in A) = p; \quad P(a \in B | a \in A) = \rho_1; \quad P(a \in B | a \notin A) = \rho_2$$

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Let $\vec{\rho} = (p, \rho_1, \rho_2)$. We call a pair of subsets selected by this process a $\vec{\rho}$-*correlated pair*. Note that when $(\rho_1, \rho_2) = (1, 0)$, this is the old case of (A, A). When $(\rho_1, \rho_2) = (0, 1)$, this is (A, A^c). When $\rho_1 = \rho_2$, A and B are independent.

Correlated Random Pairs

Let $P(\vec{\rho}, n)$ be the probability that a $\vec{\rho}$-correlated pair (A, B) with $A, B \subset \{0, \dots, n\}$ is MSTD, that is

$$|A + B| > |\pm(A - B)| = |(A - B) \cup (B - A)|$$

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Theorem

For any $\vec{\rho} \in [0, 1]^3$, the limit

$$\lim_{n \to \infty} P(\vec{\rho}, n) =: P(\vec{\rho})$$

exists. Moreover, as long as $p \notin \{0, 1\}$ and $(\rho_1, \rho_2) \neq (0, 0), (1, 1)$, then $P(\vec{\rho})$ is strictly positive.

The function $P(\vec{\rho})$

Theorem

The function $P(\vec{\rho})$ is continuous on $[0,1]^3$.

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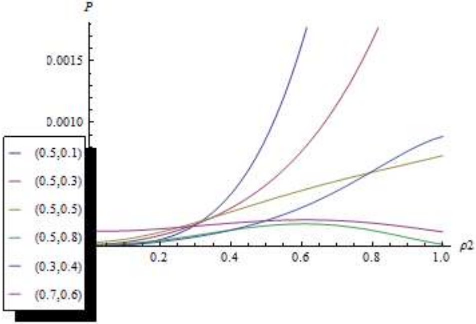
Here we fix $n = 9$ and investigate how the percentage P changes when we vary p, ρ_1, ρ_2 and see where it is maximized.

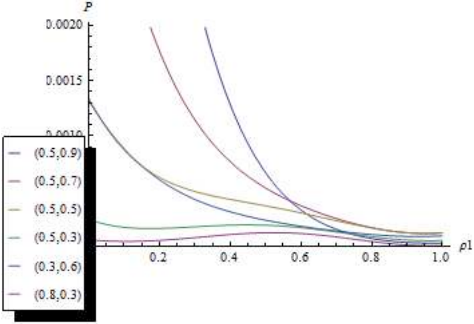
We find the maximum percentage at point $(0.5, 0, 1)$.

Fix (p, ρ_1)



Conjecture 1: For any fixed (p, ρ_1) with ρ_1 not too big ($\rho_1 \leq .4$) then P as a function of ρ_2 is strictly increasing in $[0, 1]$ and reaches its maximum at $\rho_2 = 1$.

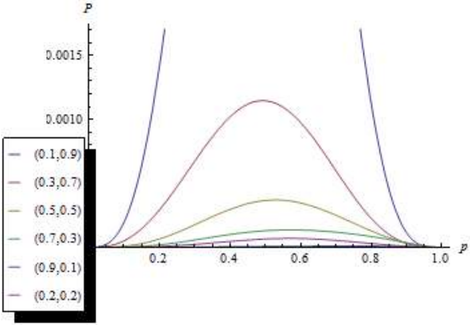
Fix (p, ρ_2)



Conjecture 2: For any fixed (p, ρ_2) with ρ_2 not too small $(\rho_2 \geq .5)$ then P as a function of ρ_1 is strictly decreasing in $[0, 1]$ and reaches its maximum at $\rho_1 = 0$.

Fix (ρ_1, ρ_2)

$n = 9$: If we fix (ρ_1, ρ_2), P as a function of p has a shape similar to parabola with a maximum at a point around $1/2$.



Conjecture 3: The maximum of function $P(p, \rho_1, \rho_2)$ is at $P(1/2, 0, 1) \approx 0.03$.

The minimal MSTD pair

Hegarty (2007) proved the smallest MSTD set has size 8.

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Theorem

The smallest MSTD pair has size $(3, 5)$ or $(4, 4)$.

Examples:

$$A = \{1, 2, 5, 7\}, \quad B = \{1, 3, 6, 7\}$$
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Distribution of Summands in Generalized Zeckendorf Decompositions

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(with Thao Do and David Moon)

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Zeckendorf's Theorem

- Our research is inspired by an elegant theorem of Zeckendorf.

Theorem

Write the Fibonacci numbers as $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. All natural numbers can be uniquely written as a sum of non-consecutive Fibonacci numbers.

Example

$$2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4$$

Going the other way

Previous work:
linear recurrence sequence $\rightarrow$ notion of legal decomposition.

Our work:
notion of legal decompostition $\rightarrow$ linear recurrence sequence.

f-Decompositions

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- We focused on constructing sequences from notions of legal decomposition.
- Many notions of "legal" decompositions can be encoded as f-decompositions.

Definition

Let $f : \mathbb{N} \to \mathbb{N}$. A sum $\sum_{i=0}^{k} a_{n_i}$ of terms of $\{a_n\}$ is a *legal f-decomposition using* $\{a_n\}$ if for every a_{n_i}, the previous $f(n_i)$ terms are not in the f-decomposition.

Result

Theorem

If $f(n+1) \leq 1 + f(n)$ for all $n \in \mathbb{N}$, there exists a sequence $\{a_n\}$ such that every positive integer has a unique legal f-decomposition using $\{a_n\}$.

Example

The Zeckendorf condition is that consecutive terms may not be chosen.
This is equivalent to saying $f(n) = 1$ for all $n \in \mathbb{N}$.
This condition yields the Fibonacci numbers.

$$\{F_n\} = 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Bins

- Base b representation can be interpretted as f-decompositions. For example, consider base 5:

$$\{a_n\} = \underbrace{1, 2, 3, 4}, \underbrace{5, 10, 15, 20}, \underbrace{25, 50, 75, 100}, \dots$$

Here, $a_n = 5a_{n-4}$.

- We build upon this notion of legal decomposition by adding the Zeckendorf condition. A legal decomposition is one that contains no consecutive terms and at most one term from each bin.

$$\{a_n\} = \underbrace{1, 2, 3}, \underbrace{4, 7, 11}, \underbrace{15, 26, 41}, \underbrace{56, 97, 153}, \underbrace{209, 362, 571}, \dots$$

Here, $a_n = 4a_{n-3} - a_{n-6}$. We analyze this case in detail.

Number of summands

Our goal is to show that the number of summands for integers in $[0, a_{bn})$ converges to the Gaussian distribution as $n \to \infty$.

- Let $p_{n,k}$ be the number of integers that can be legally written as the sum of exactly k summands from n bins.

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- Define $g_n(y) = \sum_{k \geq 0} p_{n,k} y^k$. We were able to show that

$$g_n(y) = \frac{\left(by+1+\sqrt{(b^2-4)y^2+2by+1}\right)^{n+1} - \left(by+1-\sqrt{(b^2-4)y^2+2by+1}\right)^{n+1}}{2^{n+1}\sqrt{(b^2-4)y^2+2by+1}}$$

Mean and Variance

- We can use $g_n(y) = \sum_{k\geq 0} p_{n,k} y^k$ to compute mean and variance of the random variable X_n, the number of summands for integers in $[a_0, a_{bn})$.

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- The mean, μ_n, is simply

$$\mu_n = \frac{g_n'(1)}{g_n(1)} = \frac{\left(b^2+b-4+b\sqrt{b^2+2b-3}\right)}{\sqrt{b^2+2b-3}\left(1+b+\sqrt{b^2+2b-3}\right)} n + O(1)$$

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- The variance is

$$\sigma_n^2 = \frac{\frac{d}{dy}\left[y g_n'(y)\right]\Big|_{y=1}}{g(1)} - \mu^2 = \frac{(b^2+b-4)n}{(b^2+2b-3)^{3/2}} + O(1)$$

Moment Generating Function

- If we normalize X_n to $Y_n = (X_n - \mu_n)/\sigma_n$, the moment generating function of Y_n is

$$M_{Y_n}(t) = \mathbb{E}(e^{tY_n}) = \sum_{k\geq 0} \frac{p_{n,k}e^{\frac{t(k-\mu_n)}{\sigma_n}}}{\sum_{k\geq 0} p_{n,k}} = \frac{g_n(e^{t/\sigma_n})e^{-t\mu_n/\sigma_n}}{g_n(1)}$$

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- After multiple Taylor Series expansions, we get

$$\log(M_{Y_n}(t)) = \frac{t^2}{2} + O\left(\frac{t^3}{\sqrt{n}}\right)$$

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- Hence, the distribution of Y_n converges to the standard normal distribution.

Far-Difference Representations

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- Alpert [Al] proved that a Fibonacci far-difference representation exists for all integers and is unique.
- Miller-Wang [MW], a paper from a previous SMALL summer with countably infinite pages, proved Gaussianity for Alpert's far-difference representations.

Preliminary Definitions

The k-Skipponaccis are recurrence relations of the form

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Alpert [Al] proved the following result for the Fibonaccis (also called the 1-Skipponaccis).

Alpert's Theorem

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the Fibonaccis such that all terms of the same sign are at least 4 apart in index, and all terms of opposite sign are at least 3 apart in index.

Our First Result

Example: $119 = 144 - 34 + 8 + 1 = \underbrace{F_{11} - F_8}_{\text{3 apart}} + \underbrace{F_5 + F_1}_{\text{4 apart}}$

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Theorem 1

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the k-Skipponaccis such that all terms of the same sign are at least $2k + 2$ apart in index and all terms of opposite sign are at least $k + 2$ apart in index.

Our Second Result

$$\text{Let } R(n) = \sum_{0 < n - b(2k+2) \leq n} S_{n-b(2k+2)} = S_n + S_{n-2k-2} + S_{n-4k-4} + \dots$$

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Theorem 2

Let $\mathcal{K}_n$ and $\mathcal{L}_n$ be random variables denoting the number of positive and negative summands in the far-difference representation of integers on the interval $(R_{n-1}, R_n]$. As $n \to \infty$, the joint density of $\mathcal{K}_n$ and $\mathcal{L}_n$ converges to a bivariate Gaussian.

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- This theorem expands upon the range of recurrences handled by Miller-Wang [MW].

Further Research and Open Questions

- We have further generalized the k-Skipponaccis to recurrence relations of the form $S_n = S_{n-1} + S_{n-x} + S_{n-y}$, where x is the distance between same-sign summands and y is the distance between opposite-sign summands.

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- We believe it can be shown that every far-difference restriction (x, y) uniquely defines a sequence of numbers.

- We want to prove that the number of summands in *every* (x, y) far-difference representations approaches a Gaussian.

References

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Primes and L-functions

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Motivation: Random Matrix Theory

- *L*-functions are functions on the complex plane that generalize the Riemann zeta function:

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

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- Montgomery-Dyson, Katz-Sarnak: Spacing statistics of zeros match spacing statistics of angles of eigenvalues of a random matrix.

Zero Statistics

- Possible statistics: Correlated density of zeros (n-level density), distribution of spacings between zeros (pair correlation), moments along the critical line.

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$$G = \mathrm{O}(N), \mathrm{U}(N), \mathrm{USp}(2N), \mathrm{SO(even)}(2N), \mathrm{SO(odd)}(2N+1)$$

Testing Katz-Sarnak Density Conjectures

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- Often, too hard unless we assume $\operatorname{supp}\hat{\phi} \subseteq [-\sigma, +\sigma]$ for some fixed σ.
- Need to estimate very hard sums over primes to increase the support.

Iwaniec, Luo, and Sarnak (ILS)

- Studied family of cuspidal weight-k, level-N holomorphic newforms as $N \to \infty$.

Iwaniec, Luo, and Sarnak (ILS)

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- By getting support in the range $(-2, +2)$, able to distinguish L-functions according to sign.
- To extend support of test function beyond $(-2, +2)$, need to assume a conjecture called Hypothesis S.

Hypothesis S

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Let c be a positive integer, and let a be coprime to c. Then for $A \geq 0$ and some $\alpha \in [1/2, 3/4)$, we have

$$\sum_{\substack{p \leq X \\ p \equiv a \bmod c}} e^{2\pi i \left(\frac{2\sqrt{p}}{c}\right)} \ll c^A X^\alpha.$$

- Remark: $f(x) \ll g(x)$ means that $|f(x)| \leq k|g(x)|$ for some k and for all $x \geq x_0$.

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- Remark: $f(x) \ll g(x)$ means that $|f(x)| \leq k|g(x)|$ for some k and for all $x \geq x_0$.
- Question: Why should we expect this sum to be smaller than X?

Philosophy of "Square-Root Cancellation"

- Take sequence of real numbers $a_1, \ldots, a_N$.

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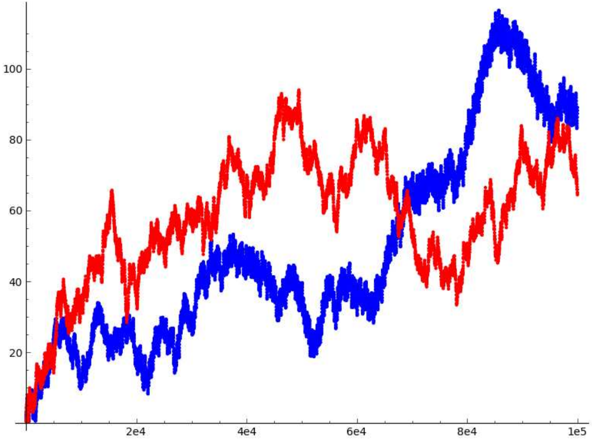
- Expect similar thing to happen for Hypothesis S.

Pretty Picture



- Sum over Primes, Sum over Random Real Numbers

Cracking Hypothesis S

- Plan: relate information about primes to information about zeros of L-functions.

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- When $c = 1$ the L-function is Riemann zeta function; same ideas work for all c.
- Assume Riemann Hypothesis to get control over zeros.

First Steps

- Do fancy tricks to make sum nicer to work with.

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Riemann-von Mangoldt Explicit Formula

Let Λ be the von Mangoldt function (like indicator for prime powers). Then

$$\sum_{n\leq x}\Lambda(n) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} + O\left(\frac{x\log^2(xT)}{T} + \log x\right),$$

where sum is over zeros of Riemann zeta function.

It's (Probably) True

Theorem (A-C-M-P-T)

Assume RH and that $\frac{2}{\pi e}\gamma_n \log(\frac{\gamma_n}{2\pi e})$ *is well distributed mod 1 (*γ_n *is ordinate of nth zeta zero). Then Hypothesis S is true for* $c = 1$ *with* $\alpha = 3/4 - \epsilon$ *with* ϵ *very small. That is,*

$$\sum_{p \leq X} e^{2\pi i(2\sqrt{p})} \ll X^{3/4-\epsilon}.$$

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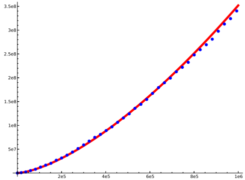
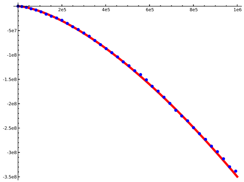
- Expect sum to have a **main term** (not just being bounded) to give agreement with random matrix theory
- Difficult to even find conjecture about what to prove; working on this

More Pretty Pictures





- Real and imaginary parts of

$$\sum_{p_1, p_2 \leq X} e^{2\pi i \cdot 2\sqrt{p_1 p_2}} \qquad \text{(here, } X_1 = X_2 = X\text{)}$$

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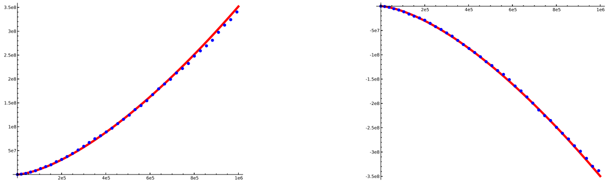

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- Not as random as the one-dimensional sum!

- Thank You!
- Next up, we have . . .

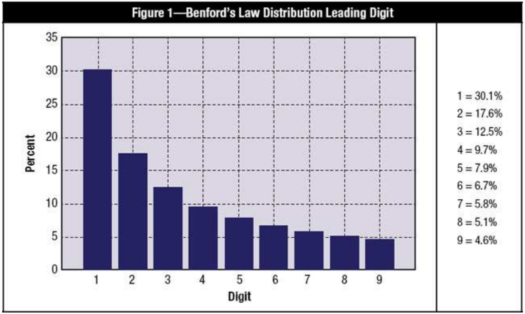
Benford Behavior of Dependent Random Variables

Taylor Corcoran - University of Arizona
Jaclyn Porfilio - Williams College
Jirapat Samranvedhya - Williams College

Benford's Law

Definition

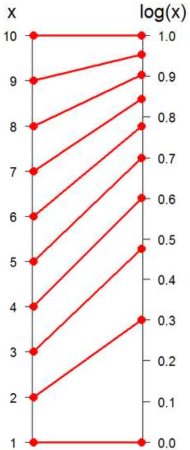
A dataset is said to follow **Benford's Law** (base b) if the probability of observing a first digit of d is $\log_b \frac{1+d}{d}$.



Logarithms and Benford's Law

$\mathbb{P}(\text{leading digit } d) = \log(d+1) - \log(d)$

Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Stick Decomposition

Fixed Proportion Decomposition Process

Decomposition Process

1. Consider a stick of length $\mathcal{L}$.

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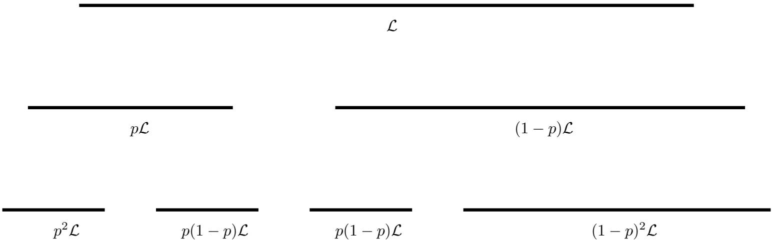
4. Repeat N times (using the same proportion).

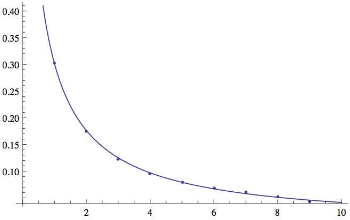
Fixed Proportion Decomposition Process



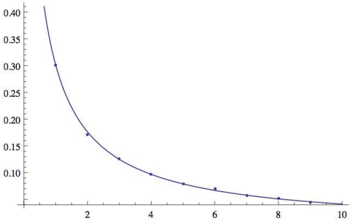
Fixed Proportion Conjecture

Joy Jing's Conjecture

The above decomposition process results in stick lengths that obey Benford's Law as $N \to \infty$ for any $p \in (0,1)$, $p \neq \frac{1}{2}$.



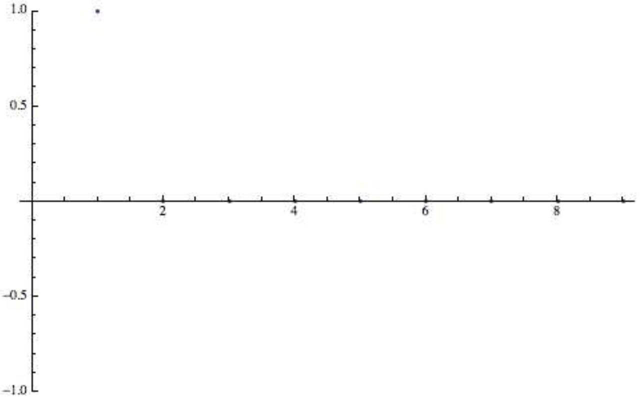
(B) $p = 0.51$ and $N = 10000$.



(B) $p = 0.99$ and $N = 50000$. Benford distribution overlaid.

Counterexample: $p = \frac{1}{11},\ 1 - p = \frac{10}{11}$.

```
BenfordFixedCut2[1 / 11, Floor[SetAccuracy[Random[], 50] * 10 ^ 4]]
```



Benford Analysis

After Nth interation,

- 2^N sticks
- $N + 1$ distinct lengths.

Distinct lengths are given by

$$x_{j+1} = \left(\frac{1-p}{p}\right) x_j, \;\; x_0 = p^N.$$

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Let $\frac{1-p}{p} = 10^y$.

$\frac{1-p}{p} = 10^y$, y $\in \mathbb{Q}$

Theorem

Let $\frac{1-p}{p} = 10^y$. If $y \in \mathbb{Q}$, the described decomposition process results in stick lengths that do not obey Benford's Law.

Let $y = \frac{r}{q}$.

Leading digit of x_j repeats every q indices. Thus,

$$\sum_k P(x_{j+kq}) = \sum_k \binom{N}{j+kq}.$$

Series Multisection

Multisection Formula

If $f = \sum_{n=-\infty}^{\infty} a_n x^n$,

$$\sum_{k=-\infty}^{\infty} a_{kq+j} x^{kq+j} = \frac{1}{q} \sum_{p=0}^{q-1} \omega^{-jp} f(\omega^p x)$$

where ω is the primitive qth root of unity $e^{2\pi i/q}$.

$\frac{1-p}{p} = 10^y,\ y \in \mathbb{Q}$

$$\sum_k P(x_{j+kq}) = \frac{1}{q}\left(1 + \mathscr{E}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right)$$

$\frac{1-p}{p} = 10^y$, $y \in \mathbb{Q}$

$$\sum_k P(x_{j+kq}) = \frac{1}{q}\left(1 + \mathscr{E}\left[(q-1)\left(\cos\frac{\pi}{q}\right)^N\right]\right)$$

Digit frequencies are multiples of $\frac{1}{q}$.

Benford frequencies are irrational, so *not* perfect Benford.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Outline

Theorem

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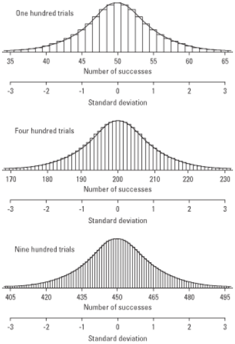
$\{x_j\} \sim Bin(N, \frac{1}{2})$
mean: $\frac{N}{2}$
standard deviation: $\frac{\sqrt{N}}{2}$

Outline of proof strategy:

1. Truncation
2. Break into intervals
   - Roughly equal probability
   - Equidistribution



$\frac{1-p}{p} = 10^y$, $y \notin \mathbb{Q}$: Truncation

For $\epsilon > 0$, Chebyshev's Inequality gives

$$\begin{aligned} P\left(\left|x - \frac{N}{2}\right| \geq N^{\frac{1}{2}+\epsilon}\right) &= P\left(\left|x - \frac{N}{2}\right| \geq N^{\epsilon} N^{\frac{1}{2}}\right) \\ &\leq \frac{1}{N^{2\epsilon}}. \end{aligned}$$

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So we can limit our analysis to

- One standard deviation
- Right half of binomial

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Intervals and Roughly Equal Probability

$\mathcal{I}_\ell = \{x_\ell, x_\ell + 1, \dots, x_\ell + N^\delta - 1\}$.

Let $x_0 = N/2$. It follows that $x_\ell = N/2 + \ell N^\delta$.

$$\left| \binom{N}{x_\ell} - \binom{N}{x_{\ell+1}} \right| \leq \binom{N}{x_\ell} N^{-\frac{1}{2}+\delta+\epsilon},$$

when $\delta < 1/2 - \epsilon$ and $\ell \leq N^{1/2-\delta+\epsilon}$.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Definition

$\{x_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if for any $[a,b] \subset [0,1]$, $P\left(x_n \bmod 1 \in [a,b]\right) \to b-a$:

$$\lim_{N\to\infty} \frac{\#\{n \le N : x_n \bmod 1 \in [a,b]\}}{N} = b-a.$$

Recall: Leading digits of stick lengths are Benford if their logarithms are equidistributed modulo 1.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Consider an interval I_ℓ where

$$I_\ell = \{x_\ell + i : i \in \{0, 1, \dots, N^\delta - 1\}\}$$

$$J_\ell \subset \{0, 1, \dots, N^\delta - 1\} = \{i : \log(x_\ell + i) \bmod 1 \in [a, b]\}.$$

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If the irrationality exponent κ of y is finite,

$$|J_\ell| \quad = \quad (b-a)N^\delta + O(N^{\delta(1-\frac{1}{\kappa}+\epsilon')})$$

$\frac{1-p}{p} = 10^y$, $y \notin \mathbb{Q}$: Equidistribution

Using

- equidistribution within intervals
- roughly equal probability

we have

$$\sum_{\ell} \sum_{i \in J_\ell} f(x_\ell + i) \quad = \quad (b-a) + O(N^{\delta(-\frac{1}{\kappa}+\epsilon')} + N^{-\frac{1}{2}+\delta+\epsilon}).$$

where κ is the irrationality exponent of y.

$\frac{1-p}{p} = 10^y$, y $\notin \mathbb{Q}$: Equidistribution

Remark

- Rate of convergence depends on $\kappa < \infty$
- Still Benford for $\kappa = \infty$, but no quantified rate.

Additive Stick Decomposition Processes: Conjectures

Benford

- Stop at evens (proved)
- Stop at primes
- Cutting into *m* pieces

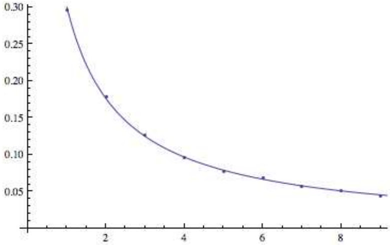
Non-Benford

- Stop at squares
- Stop at powers of two
- Stop at powers of three
- Stop at Fibonnaci numbers

Any Questions?

Additive Decomposition Process (Evens Model)

```
Digit Count After 23127 iterations: {6858, 4137, 2941, 2241, 1796, 1587, 1331, 1205, 1032}
Benford Prob for d = 1 is 0.30103 and observe 0.296524.
Benford Prob for d = 2 is 0.176091 and observe 0.178874.
Benford Prob for d = 3 is 0.124939 and observe 0.127162.
Benford Prob for d = 4 is 0.09691 and observe 0.0968955.
Benford Prob for d = 5 is 0.0791812 and observe 0.0776548.
Benford Prob for d = 6 is 0.0669468 and observe 0.0686181.
Benford Prob for d = 7 is 0.0579919 and observe 0.0575493.
Benford Prob for d = 8 is 0.0511525 and observe 0.0521013.
Benford Prob for d = 9 is 0.0457575 and observe 0.0446212.
There were 1 pieces of length one.
There were 23128 fragmented pieces.
The value of Chi Squared (goodness of fit for Benford) is 6.27561.
```



Irrationality Exponent

Let $x \in \mathbb{R}$. Denote by $\mathcal{A}$ the set of positive numbers n for which

$$0 \leq \left| x - \frac{p}{q} \right| \leq \frac{1}{q^n}$$

has at most finitely many solutions for $p, q \in \mathbb{Z}$.
The irrationality measure of x, denoted $n(x)$, is $\inf_{n \in \mathcal{A}} n$.
If $\mathcal{A}$ is empty, $n(x) = \infty$
For nonempty $\mathcal{A}$,

$$n(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 2 \text{ if } x \text{ is algebraic of degree } > 1 \\ \geq 2 \text{ if } x \text{ is transcendental} \end{cases}$$

Acknowledgements

We would like to thank our advisor, Steven J. Miller, our co-author Joseph R. Iafrate, as well as Frederick W. Strauch and Joy Jing.

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