A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior

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Integer Decompositions and Why We Care

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Motivating Question

Are there other space-efficient ways of decomposing the integers into unique collections of summands?

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Every integer can be written uniquely as a sum of non-consecutive Fibonacci numbers of the form $F_{n+1} = F_n + F_{n-1}$ (called the Zeckendorf decomposition).

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Example 1: $65 = 55 + 8 + 2 = F_9 + F_5 + F_2$

Example 2: $65 = 34 + 21 + 8 + 2 = F_8 + F_7 + F_5 + F_2$

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- Alpert [Al] proved that a Fibonacci far-difference representation exists for all integers and is unique.
- Miller-Wang [MW] extended the results for PLRS to Alpert's far-difference representations.

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- We generalize the Fibonacci recurrence to a collection of sequences called the k-Skipponaccis.
- We find unique far-difference representations using our Skipponacci sequences.
- We prove Gaussianity for every far-difference representation.

Preliminary Definitions and Results

The k-Skipponaccis are recurrence relations of the form

$$S_{n+1} = S_n + S_{n-k}$$

for some $k \geq 0$ and initial terms $1, 2, 3, \ldots, k-1, k$.

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Alpert [Al] proved the following result for the Fibonaccis (also called the 1-Skipponaccis).

Alpert's Theorem

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the Fibonaccis such that all terms of the same sign are at least 4 apart in index, and all terms of opposite sign are at least 3 apart in index.

Our First Result

Example: $119 = 144 - 34 + 8 + 1 = \underbrace{F_{11} - F_8}_{\text{3 apart}} + \underbrace{F_5 + F_1}_{\text{4 apart}}$

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Theorem 1

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the k-Skipponnacis such that all terms of the same sign are at least $2k + 2$ apart in index and all terms of opposite sign are at least $k + 2$ apart in index.

Outline of Proof

Let $R(n) = \sum_{0<n-b(2k+2)\leq n} S_{n-b(2k+2)} = S_n + S_{n-2k-2} + S_{n-4k-4} + \dots$

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We first partition the integers into intervals of the form:
$[S_n - R(n-k-2), S_n + R(n-2k-2)]$.

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For the inductive step, assume that all integers on $[0, R(n-1)]$ have a unique far-difference representation.

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For the inductive step, assume that all integers on $[0, R(n-1)]$ have a unique far-difference representation.

Next, for an integer x on the above interval, we have:

- If $x < S_n$, then $0 \leq S_n - x \leq S_n - R(n-k-2)$.
- If $x > S_n$, then $0 \leq x - S_n \leq S_n - R(n-2k-2)$.

Outline of Proof

By induction, we show that:

- $x - S_n$ has a unique decomposition with main term at most S_{n-2k-2}
- $S_n - x$ has a unique decomposition with main term at most S_{n-k-2}

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Adding S_n to the representations above gives us:

- $S_n + S_{n-2k-2} + \dots$
- $S_n - S_{n-k-2} + \dots$

which are both legal decompositions. □

Distribution of number of summands

- We have shown that, on the interval $(R_n, R_{n+1}]$, each integer x has an unique representation $x = \sum_{i=1}^{m} S_{n_i} - \sum_{i=1}^{\ell} S_{k_i}$, where x has m positive summands and ℓ negative summands in its far-difference representation.

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- We are interested in the distribution of the number of positive and negative summands when considering all integers in a single interval.
- We prove this distribution converges to a Gaussian (normal) distribution when $n \to \infty$.

Recurrence Relation

- Let $p_{n,m,\ell}$ denote the number of numbers in the interval $(R_n, R_{n+1}]$ whose decomposition consists of m positive summands and ℓ negative summands.

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- We derive the following recurrence relation for the number of positive and negative summands:

Recurrence relation of $p_{n,m,\ell}$

$$
\begin{aligned}
p_{n,m,\ell} &= 2p_{n-1,m,\ell} - p_{n-2,m,\ell} + p_{n-(2k+2),m-1,\ell} + p_{n-(2k+2),m,\ell-1} \\
&+ p_{n-(2k+3),m-1,\ell} - p_{n-(2k+3),m,\ell-1} + p_{n-(2k+4),m-1,\ell-1} \\
&- p_{n-(4k+4),m-1,\ell-1}
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Generalized Generating function

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Consider the product $P(x, y, z)G(x, y, z)$, where $P(x, y, z)$ is the characteristic polynomial of the recurrence relation given by:

$$P(x, y, z) = 1 - 2z + z^2 - (x + y)\left(z^{2k+2} + z^{2k+3}\right) - xy\left(z^{2k+4} + z^{4k+4}\right)$$

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Key Insight: For any $p_{n,m,\ell}$ that satisfies the recurrence relation, the coefficient in the above product will be zero.

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Key Insight: For any $p_{n,m,\ell}$ that satisfies the recurrence relation, the coefficient in the above product will be zero.

The only terms that remain are:

$$P(x, y, z)G(x, y, z) = xz - xz^2 + xyz^{k+3} - xyz^{2k+3}$$

Generalized Generating Function

Now solving for $G(x, y, z)$ gives us:

Generating function of $p_{n,m,l}$

$$G(x, y, z) = \frac{xz - xz^2 + xyz^{k+3} - xyz^{2k+3}}{1 - 2z + z^2 - (x + y)\left(z^{2k+2} + z^{2k+3}\right) - xy\left(z^{2k+4} + z^{4k+4}\right)}$$

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It will be useful to factor out $(z - 1)$ from $G(x, y, z)$, which gives us the following modified generating function:

$$G(x,y,z) = \frac{xz + xy\sum_{j=k+3}^{2k+2} z^j}{1 - z - (x+y)z^{2k+2} - xy\sum_{j=2k+4}^{4k+3} z^j}$$

Using $G(x, y, z)$ to Prove Gaussian Behavior

- Let $\mathcal{K}_n$ and $\mathcal{L}_n$ be the corresponding random variables denoting the number of positive summands and the number of negative summands in the far-difference representation for integers in $(R_{n-1}, R_n]$.

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- We prove that for any $a, b \geq 0$, the random variable $X_n = a\mathcal{K}_n + b\mathcal{L}_n$ converges to the Gaussian distribution as $n \to \infty$

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- This can be achieved by proving that every moment of $(X_n - \mathbb{E}[X_n])/\sigma_n$ approaches the corresponding moment of the standard normal $N(0, 1)$.

Calculating Moments

- For a fixed n consider $\hat{g}(x, y) = \sum p_{n,m,l} x^m y^l$. Letting $x = w^a, y = w^b$, we get $g(w) = \sum p_{n,m,l} w^{am+bl}$.

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- The range of X_n is $g(1) = \sum_{m,l \geq 0} p_{n,m,l} = R_{n+1} - R_n$ and since $g'(w) = \sum (am + bl) p_{n,m,l} w^{am+bl-1}$, we get:

$$\mathbb{E}[X_n] = g'(1)/g(1) = \mu_n.$$

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- To find the other moments, we construct a sequence of functions $g_j(w)$ such that $g_{j+1} = (xg_j(x))'$. By induction the m^{th} moment of $X_n - \mu_n$, denoted $\mu_n(m)$, is calculated as $g_m(1)/g(1)$.

Useful Result from Miller-Wang [MW]

We have Gaussianity if we can prove that each moment tends to that of the normal, or equivalently:

Theorem 2 (Miller-Wang)

For any integer $u \geq 1$:

$$\frac{\mu_n(2u-1)}{\sigma_n^{2u-1}} \to 0 \quad \text{and} \quad \frac{\mu_n(2u)}{\sigma_n^{2u}} \to (2u-1)!! \quad \text{as} \quad u \to \infty$$

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- Fortunately for us, Miller-Wang proved this theorem for a large subset of recurrence relations.
- Thus, if we can prove the conditions specified by Miller-Wang, we can extend these results to handle a greater range of recurrences.

Conditions for Gaussianity

- We factor out the denominator P our generating function and use partial fractions to get an explicit formula for $g(w)$, namely $\sum_{i=1}^{4k+3} wq_i(w)/e_i^n(w)$ where $e_i(w)$ are roots of P.

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- We want to reduce $g(w)$ to $wq_1(w)/e_1^n(w)$ so that it behaves similarly to an n-power function. It is enough to prove that P has no multiple roots and a single root whose norm is strictly less than that of all others.

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- We also want a formula for $e_i'(w)$ to prove that the mean and variance grow linearly with n.

Proposition 1

The following captures *most* of the necessary conditions.

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(b) There exists exactly one positive real root $e_1(w)$ such that $e_1(w) < 1$ and $e_1(w) < |e_i(w)|$ for all $i \geq 2$.

(c) Each root $e_i(w)$ is continuous and ℓ-times differentiable for any $\ell \geq 1$ and

$$e_i'(w) = -\frac{(aw^{a-1}+bw^{b-1})e_i(w)^{2k+2}+(a+b)w^{a+b-1}\sum_{j=2k+4}^{4k+3} e_1(w)^j}{1+(w^a+w^b)(2k+2)e_i(w)^{2k+1}+w^{a+b}\sum_{j=2k+4}^{4k+3} je_i(w)^{j-1}}$$

Proof of (a)

Let $A_w(z)$ be the denominator of our generating function. We will consider only the case where $w = 1$. This gives us:

$$A(z) = 1 - 2z + z^2 - 2z^{2k+2} + 2z^{2k+3} - z^{2k+4} + z^{4k+4},$$

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We prove that no roots are repeated by:

- Showing that the factors are pairwise co-prime.
- Showing that the gcd of each factor and its derivative is 1.

Proof of (b)

Let $\hat{A}_w(z)$ be the denominator of our revised generating function. We once again consider the case where $w = 1$, which gives us:

$$\hat{A}(z) = 1 - z - 2z^{2k+2} - \sum_{j=2k+4}^{4k+3} z^j$$

$$\hat{A}'(z) = -1 - 2(2k+2)z^{2k+1} - \sum_{j=2k+4}^{4k+3} jz^{j-1}$$

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- Note that $\hat{A}(0) > 0$, while $\hat{A}(1) < 0$, so there must be a root e_1 on (0,1)
- Moreover, since $\hat{A}'(z)$ is negative for all $z \geq 0$, the function is strictly decreasing on $(0, \infty)$. Thus e_1 is unique.

Proof of (b)

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$$1 = \left| e_i + 2e_i^{2k+2} + \sum_{j=2k+4}^{4k+3} e_i^j \right| \leq |e_i| + 2|e_i|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_i|^j$$

$$\leq |e_1| + 2|e_1|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_1|^j = e_1 + 2e_1^{2k+2} + \sum_{j=2k+4}^{4k+3} e_1^j = 1$$

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Thus equality must hold everywhere, but this contradicts (a). It follows that $e_1 < |e_i|$. □

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&= [1 - e_1(w) - (w^a + w^b)e_1(w)^{2k+2} - w^{a+b}\sum_{j=2k+4}^{4k+3} e_1(w)^j] - \\
&\quad [1 - e_1(w+\Delta w) - ((w+\Delta w)^a + (w+\Delta w)^b)e_1(w+\Delta w)^{2k+2} \\
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&\quad [1 - e_1(w+\Delta w) - ((w+\Delta w)^a + (w+\Delta w)^b)e_1(w+\Delta w)^{2k+2} \\
&\quad - (w+\Delta w)^{a+b}\textstyle\sum_{j=2k+4}^{4k+3} e_1(w+\Delta w)^j] \\
&= e_1(w+\Delta w) - e_1(w) + (w^a + w^b)[e_1(w+\Delta w)^{2k+2} - e_1(w)^{2k+2} \\
&\quad + w^{a+b}\textstyle\sum_{j=2k+4}^{4k+3}[e_1(w+\Delta w)^j - e_1(w)^j] \\
&\quad + e_1(w+\Delta w)^{2k+2}[(w+\Delta w)^a - w^a + (w+\Delta w)^b - w^b] \\
&\quad + \left[\textstyle\sum_{j=2k+4}^{4k+3} e_1(w+\Delta w)^j\right]\left[(w+\Delta w)^{a+b} - w^{a+b}\right]
\end{aligned}$$

Proof of (c)

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$$\begin{aligned} 0 \quad = \quad & \left[e_1(w+\Delta w)-e_1(w)\right]\Big(1+(w^a+w^b)\textstyle\sum_{i=0}^{2k+1} e_1(w+\Delta w)^i e_1(w)^{2k+1-i} \\ & +w^{a+b}\textstyle\sum_{j=2k+4}^{4k+3}\sum_{i=0}^{j-1} e_1(w+\Delta w)^i e_1(w)^{j-1-i}\Big) \\ & +\Delta w\Big(e_1(w+\Delta w)^{2k+2}[\textstyle\sum_{i=0}^{a-1}(w+\Delta w)^i w^{a-1-i}+\sum_{i=0}^{b-1}(w+\Delta w)^i w^{b-1-i}] \\ & +\big[\textstyle\sum_{j=2k+4}^{4k+3} e_1(w+\Delta w)^j\big]\sum_{i=0}^{a+b-1}(w+\Delta w)^i w^{a+b-1-i}\Big) \end{aligned}$$

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Now, since $e_i(w)$ is continuous, and w^a, w^b, and w^{a+b} are differentiable at $w=1$, we can rearrange terms and take the limit of the above equation, which gives us:

$$\begin{aligned} e_1'(w) &= \lim_{\Delta w\to 0}\frac{e_1(w+\Delta w)-e_1(w)}{\Delta w} \\ &= -\frac{(aw^{a-1}+bw^{b-1})e_1(w)^{2k+2}+(a+b)w^{a+b-1}\sum_{j=2k+4}^{4k+3} e_1(w)^j}{1+(w^a+w^b)(2k+2)e_1(w)^{2k+1}+w^{a+b}\sum_{j=2k+4}^{4k+3} je_1(w)^{j-1}} \qquad \square \end{aligned}$$

Home Stretch

We now seek an expression for the coefficients of z^n in our generating function. As before, will denote this function as:

$$g(w) = \sum_{m>0, l\geq 0} p_{n,m,l} w^{am+b\ell}$$

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For our calculations, it will be convenient to express the denominator of our generating function in terms of it's partial fraction expansion.

$$\frac{1}{A_w(z)} = \frac{1}{w^{a+b}} \sum_{i=1}^{4k+3} \frac{1}{(z - e_i(w)) \prod_{j\neq i}(e_j(w) - e_i(w))}$$

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We rearrange this formula to get:

$$\frac{1}{A_w(z)} = \frac{1}{w^{a+b}} \sum_{i=1}^{4k+3} \frac{1}{(1 - \frac{z}{e_i(w)})} \cdot \frac{1}{e_i(w) \prod_{j \neq i}(e_j(w) - e_i(w))}$$

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Key Insight: $\frac{1}{1-\frac{z}{e_i(w)}}$ represents a geometric series.

We now sum over all coefficients of z^n in the numerator of our generating function, which gives us:

$$g(w) = \sum_{i=1}^{4k+3} \frac{w^{-b}(1 - e_i(w)) + e^{k+2}(w) - e^{2k+2}(w)}{e_i^n(w) \prod_{j \neq i}(e_j(w) - e_i(w))}$$

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If we let $q_i(w)$ denote all terms of $g(w)$ that do not depend on n, then $g(w)$ reduces to $\sum_{i=1}^{4k+3} wq_i(w)/e_i^n(w)$.

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- For all $e_i(w)$ where $i \geq 2$, we have $e_1^n(w) \ll e_i^n(w)$ as $n \to \infty$.
- $g(w) = wq_1(w)/e_1^n(w) + o(\gamma_{a,b}^n)$, where $\gamma_{a,b}^n$ is dependent only on a, b and is negligible for n large.

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$$= \frac{q_1'(1)e^{2n}(1)}{e_1^{2n}(1)q_1(1)} - \frac{nq_1(1)e_1^{2n-1}(1)e_1'(1)}{e_1^{2n}(1)q_1(1)}$$
$$= \frac{q_1'(1)}{q_1(1)} - \frac{e_1'(1)}{e_1(1)}n$$

Letting $C_{a,b} = -e_1'(1)/e_1(1)$ and $d_{a,b} = p'(1)/p_1(1)$ gives us a computable expression for the mean in the form:

$$C_{a,b}n + d_{a,b} + o(\gamma_{a,b}^n)$$

Grand Finale!

Since our mean grows linearly with n, Miller-Wang [MW] gives us a way to find the variance in the form:

$$h'_{a,b}(1)n + q''_1(1) + o(\tau^n_{a,b})$$

Where $h_{a,b}(w) = \frac{we'_1(w)}{e_1(w)} - C_{a,b}$, and the constants $\tau^n_{a,b} \in (0,1)$ and $p''_1(1)$ depend only on a and b.

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- Using our proposition, it is easy to verify that $C_{a,b}$ and $h'_{a,b} \neq 0$, and thus the mean and variance are not independent of n.
- Now that we have a mean μ_n and a variance σ_n that grow in linear time, we are done! □

Recap: What Just Happened?

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3. We proved that the distribution of summands in any k-Skipponacci far-difference representation approaches a Gaussian distribution.

Further Research and Open Questions

- We have further generalized the k-Skipponacis to recurrence relations of the form $S_n = S_{n-1} + S_{n-x} + S_{n-y}$, where x is the distance between same-sign summands and y is the distance between opposite-sign summands.

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- We believe it can be shown that every far-difference restriction (x, y) uniquely defines a sequence of numbers.

- We want to prove that the number of summands in *every* (x, y) far-difference representations approaches a Gaussian.

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