A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior

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Integer Decompositions and Why We Care

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**Motivating Question**

Are there other space-efficient ways of decomposing the integers into unique collections of summands?
Zeckendorf’s Theorem (1972)

Every integer can be written uniquely as a sum of non-consecutive Fibonacci numbers of the form $F_{n+1} = F_n + F_{n-1}$ (called the Zeckendorf decomposition).
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Example 1: \( 65 = 55 + 8 + 2 = F_9 + F_5 + F_2 \)

Example 2: \( 65 = 34 + 21 + 8 + 2 = F_8 + F_7 + F_5 + F_2 \)
Far-Difference Representations

Definition

A far-difference representation is a sum of numbers and their additive inverses.
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- Alpert [Al] proved that a Fibonacci far-difference representation exists for all integers and is unique.

- Miller-Wang [MW] extended the results for PLRS to Alpert’s far-difference representations.
What’s New in Our Results?

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What’s New in Our Results?

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- We generalize the Fibonacci recurrence to a collection of sequences called the $k$-Skipponaccis.

- We find unique far-difference representations using our Skipponacci sequences.

- We prove Gaussianity for every far-difference representation.
The $k$-Skipponaccis are recurrence relations of the form

$$S_{n+1} = S_n + S_{n-k}$$

for some $k \geq 0$ and initial terms $1, 2, 3, \ldots, k-1, k$. 
The \textit{k-Skipponaccis} are recurrence relations of the form
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S_{n+1} = S_n + S_{n-k}
\]
for some $k \geq 0$ and initial terms $1, 2, 3, \ldots, k-1, k$.

Alpert [Al] proved the following result for the Fibonaccis (also called the 1-Skipponaccis).

\textbf{Alpert’s Theorem}

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the Fibonaccis such that all terms of the same sign are at least 4 apart in index, and all terms of opposite sign are at least 3 apart in index.
Our First Result

Example: $119 = 144 - 34 + 8 + 1 = \underbrace{F_{11} - F_8}_{3 \text{ apart}} + \underbrace{F_5 + F_1}_{4 \text{ apart}}$
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Example: $119 = 144 - 34 + 8 + 1 = F_{11} - F_8 + F_5 + F_1$

\[ F_{11} \quad \text{3 apart} \quad F_8 \quad \text{4 apart} \]

**Theorem 1**

Every $x \in \mathbb{Z}$ has a unique far-difference representation for the $k$-Skipponnacis such that all terms of the same sign are at least $2k + 2$ apart in index and all terms of opposite sign are at least $k + 2$ apart in index.
Outline of Proof

Let \( R(n) = \sum_{0 < n - b(2k+2) \leq n} S_{n-b(2k+2)} = S_n + S_{n-2k-2} + S_{n-4k-4} + \ldots \)
Outline of Proof

Let \( R(n) = \sum_{0 < n - b(2k+2) \leq n} S_{n-b(2k+2)} = S_n + S_{n-2k-2} + S_{n-4k-4} + \ldots \)

We first partition the integers into intervals of the form: 
\([S_n - R(n - k - 2), S_n + R(n - 2k - 2)]\).
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For the inductive step, assume that all integers on \([0, R(n - 1)]\) 
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For the inductive step, assume that all integers on \([0, R(n - 1)]\) have a unique far-difference representation.

Next, for an integer \( x \) on the above interval, we have:

- If \( x < S_n \), then \( 0 \leq S_n - x \leq S_n - R(n - k - 2) \).
- If \( x > S_n \), then \( 0 \leq x - S_n \leq S_n - R(n - 2k - 2) \).
By induction, we show that:

- \( x - S_n \) has a unique decomposition with main term at most \( S_{n-2k-2} \)
- \( S_n - x \) has a unique decomposition with main term at most \( S_{n-k-2} \)
Outline of Proof

By induction, we show that:

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Adding $S_n$ to the representations above gives us:

- $S_n + S_{n-2k-2} + \ldots$
- $S_n - S_{n-k-2} + \ldots$

which are both legal decompositions.
We have shown that, on the interval \((R_n, R_{n+1}]\), each integer \(x\) has an unique representation \(x = \sum_{i=1}^{m} S_{n_i} - \sum_{i=1}^{\ell} S_{k_i}\), where \(x\) has \(m\) positive summands and \(\ell\) negative summands in its far-difference representation.
Distribution of number of summands

We have shown that, on the interval \((R_n, R_{n+1}]\), each integer \(x\) has an unique representation
\[x = \sum_{i=1}^{m} S_{n_i} - \sum_{i=1}^{\ell} S_{k_i},\]
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We are interested in the distribution of the number of positive and negative summands when considering all integers in a single interval.

We prove this distribution converges to a Gaussian (normal) distribution when \(n \to \infty\).
Recurrence Relation

Let $p_{n,m,\ell}$ denote the number of numbers in the interval $(R_n, R_{n+1}]$ whose decomposition consists of $m$ positive summands and $\ell$ negative summands.
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We derive the following recurrence relation for the number of positive and negative summands:

\[
p_{n,m,\ell} = 2p_{n-1,m,\ell} - p_{n-2,m,\ell} + p_{n-(2k+2),m-1,\ell} + p_{n-(2k+2),m,\ell-1} + p_{n-(2k+3),m-1,\ell} - p_{n-(2k+3),m,\ell-1} + p_{n-(2k+4),m-1,\ell-1} - p_{n-(4k+4),m-1,\ell-1}
\]
Generalized Generating function

Let $G(x, y, z) = \sum p_{n,m,\ell} x^m y^\ell z^n$ be the generating function of $p_{n,m,\ell}$. 

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Consider the product \( P(x, y, z)G(x, y, z) \), where \( P(x, y, z) \) is the characteristic polynomial of the recurrence relation given by:

\[
P(x, y, z) = 1 - 2z + z^2 - (x + y) \left( z^{2k+2} + z^{2k+3} \right) - xy \left( z^{2k+4} + z^{4k+4} \right)
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Key Insight: For any \( p_{n,m,\ell} \) that satisfies the recurrence relation, the coefficient in the above product will be zero.
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Key Insight: For any $p_{n,m,\ell}$ that satisfies the recurrence relation, the coefficient in the above product will be zero.

The only terms that remain are:

$$P(x, y, z)G(x, y, z) = xz - xz^2 + xyz^{k+3} - xyz^{2k+3}$$
Generalized Generating Function

Now solving for $G(x, y, z)$ gives us:

**Generating function of $\rho_{n,m,l}$**

\[
G(x, y, z) = \frac{xz - xz^2 + xyz^{k+3} - xyz^{2k+3}}{1 - 2z + z^2 - (x + y) (z^{2k+2} + z^{2k+3}) - xy (z^{2k+4} + z^{4k+4})}
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\]

It will be useful to factor out $(z - 1)$ from $G(x, y, z)$, which gives us the following modified generating function:

\[
G(x, y, z) = \frac{xz + xy \sum_{j=k+3}^{2k+2} z^j}{1 - z - (x + y)z^{2k+2} - xy \sum_{j=2k+4}^{4k+3} z^j}
\]
Using $G(x, y, z)$ to Prove Gaussian Behavior

Let $\mathcal{K}_n$ and $\mathcal{L}_n$ be the corresponding random variables denoting the number of positive summands and the number of negative summands in the far-difference representation for integers in $(R_{n-1}, R_n]$. 

Let $X_n$ be the random variable defined as $a\mathcal{K}_n + b\mathcal{L}_n$, where $a$ and $b$ are non-negative constants. We prove that for any $a, b \geq 0$, the random variable $X_n$ converges to the Gaussian distribution as $n \to \infty$. This can be achieved by proving that every moment of $(X_n - E[X_n])/\sigma_n$ approaches the corresponding moment of the standard normal $\mathcal{N}(0, 1)$. 


Using $G(x, y, z)$ to Prove Gaussian Behavior

Let $K_n$ and $L_n$ be the corresponding random variables denoting the number of positive summands and the number of negative summands in the far-difference representation for integers in $(R_{n-1}, R_n]$.

We prove that for any $a, b \geq 0$, the random variable $X_n = aK_n + bL_n$ converges to the Gaussian distribution as $n \to \infty$. 
Using $G(x, y, z)$ to Prove Gaussian Behavior

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- We prove that for any $a, b \geq 0$, the random variable $X_n = a\mathcal{K}_n + b\mathcal{L}_n$ converges to the Gaussian distribution as $n \to \infty$.

- This can be achieved by proving that every moment of $(X_n - \mathbb{E}[X_n]) / \sigma_n$ approaches the corresponding moment of the standard normal $N(0, 1)$. 
Calculating Moments

For a fixed $n$ consider $\hat{g}(x, y) = \sum p_{n,m,l}x^my^l$. Letting $x = w^a$, $y = w^b$, we get $g(w) = \sum p_{n,m,l}w^{am+bl}$.
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The range of $X_n$ is $g(1) = \sum_{m,l \geq 0} p_{n,m,l} = R_{n+1} - R_n$ and since $g'(w) = \sum (am + bl) p_{n,m,l} w^{am+bl-1}$, we get:

$$\mathbb{E}[X_n] = g'(1)/g(1) = \mu_n.$$
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- To find the other moments, we construct a sequence of functions $g_j(w)$ such that $g_{j+1} = (xg_j(x))'$. By induction the $m^{th}$ moment of $X_n - \mu_n$, denoted $\mu_n(m)$, is calculated as $g_m(1)/g(1)$.
Useful Result from Miller-Wang [MW]

We have Gaussianity if we can prove that each moment tends to that of the normal, or equivalently:

**Theorem 2 (Miller-Wang)**

For any integer $u \geq 1$:

$$\frac{\mu_n(2u - 1)}{\sigma_n^{2u-1}} \to 0 \quad \text{and} \quad \frac{\mu_n(2u)}{\sigma_n^{2u}} \to (2u - 1)!! \quad \text{as} \quad u \to \infty$$
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- Fortunately for us, Miller-Wang proved this theorem for a large subset of recurrence relations.
- Thus, if we can prove the conditions specified by Miller-Wang, we can extend these results to handle a greater range of recurrences.
We factor out the denominator $P$ our generating function and use partial fractions to get an explicit formula for $g(w)$, namely $\sum_{i=1}^{4k+3} wq_i(w)/e_i^n(w)$ where $e_i(w)$ are roots of $P$. 
Conditions for Gaussianity

- We factor out the denominator $P$ our generating function and use partial fractions to get an explicit formula for $g(w)$, namely $\sum_{i=1}^{4k+3} wq_i(w)/e_i^n(w)$ where $e_i(w)$ are roots of $P$.

- We want to reduce $g(w)$ to $wq_1(w)/e_1^n(w)$ so that it behaves similarly to an $n$-power function. It is enough to prove that $P$ has no multiple roots and a single root whose norm is strictly less than that of all others.
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We want to reduce $g(w)$ to $wq_1(w)/e_1^n(w)$ so that it behaves similarly to an $n$-power function. It is enough to prove that $P$ has no multiple roots and a single root whose norm is strictly less than that of all others.

We also want a formula for $e'_i(w)$ to prove that the mean and variance grow linearly with $n$. 
Proposition 1

The following captures *most* of the necessary conditions.

**Proposition 1:** There exists $\epsilon \in (0, 1)$ such that for any $w \in I_\epsilon = (1 - \epsilon, 1 + \epsilon)$:

(a) The denominator of our generating function has no multiple roots.
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**Proposition 1:** There exists $\epsilon \in (0, 1)$ such that for any $w \in I_\epsilon = (1 - \epsilon, 1 + \epsilon)$:

(a) The denominator of our generating function has no multiple roots.

(b) There exists exactly one positive real root $e_1(w)$ such that $e_1(w) < 1$ and $e_1(w) < |e_i(w)|$ for all $i \geq 2$. 
Proposition 1

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(b) There exists exactly one positive real root $e_1(w)$ such that $e_1(w) < 1$ and $e_1(w) < |e_i(w)|$ for all $i \geq 2$.

(c) Each root $e_i(w)$ is continuous and $\ell$-times differentiable for any $\ell \geq 1$ and

$$e'_i(w) = -\frac{(aw^{a-1} + bw^{b-1})e_i(w)^{2k+2} + (a+b)w^{a+b-1} \sum_{j=2k+4}^{4k+3} e_1(w)^j}{1 + (w^{a} + w^{b})(2k+2)e_i(w)^{2k+1} + wa + b \sum_{j=2k+4}^{4k+3} je_i(w)^{j-1}}$$
Let $A_w(z)$ be the denominator of our generating function. We will consider only the case where $w = 1$. This gives us:

$$A(z) = 1 - 2z + z^2 - 2z^{2k+2} + 2z^{2k+3} - z^{2k+4} + z^{4k+4},$$

which factors nicely to:
Proof of (a)

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$$A(z) = (z^{2k+2} - 1)(z^{k+1} + z - 1)(z^{k+1} - z + 1).$$
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$$A(z) = (z^{2k+2} - 1)(z^{k+1} + z - 1)(z^{k+1} - z + 1).$$

We prove that no roots are repeated by:

- Showing that the factors are pairwise co-prime.
- Showing that the gcd of each factor and its derivative is 1.
Proof of (b)

Let \( \hat{A}_w(z) \) be the denominator of our revised generating function. We once again consider the case where \( w = 1 \), which gives us:

\[
\hat{A}(z) = 1 - z - 2z^{2k+2} - \sum_{j=2k+4}^{4k+3} z^j
\]

\[
\hat{A}'(z) = -1 - 2(2k + 2)z^{2k+1} - \sum_{j=2k+4}^{4k+3} jz^{j-1}
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\]

- Note that \( \hat{A}(0) > 0 \), while \( \hat{A}(1) < 0 \), so there must be a root \( e_1 \) on \((0,1)\).
- Moreover, since \( \hat{A}'(z) \) is negative for all \( z \geq 0 \), the function is strictly decreasing on \((0, \infty)\). Thus \( e_1 \) is unique.
Proof of (b)

Let $e_i$ be another root of $\hat{A}(z)$ and assume that $|e_i| \leq e_1 < 1$. Then $|e_i| \leq |e_1| = e_1'$. 

Thus equality must hold everywhere, but this contradicts (a). It follows that $e_1 < |e_i|$. 


Proof of (b)

Let $e_i$ be another root of $\hat{A}(z)$ and assume that $|e_i| \leq e_1 < 1$. Then $|e_i|^j \leq |e_1|^j = e_1^j$.

\[
1 = \left| e_i + 2e_i^{2k+2} + \sum_{j=2k+4}^{4k+3} e_i^j \right| \leq |e_i| + 2|e_i|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_i|^j \\
\leq |e_1| + 2|e_1|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_1|^j = e_1 + 2e_1^{2k+2} + \sum_{j=2k+4}^{4k+3} e_1^j = 1
\]
Proof of (b)

Let \( e_i \) be another root of \( \hat{A}(z) \) and assume that \( |e_i| \leq e_1 < 1 \). Then \( |e_i|^j \leq |e_1|^j = e_1^j \).

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1 = \left| e_i + 2e_i^{2k+2} + \sum_{j=2k+4}^{4k+3} e_i^j \right| \leq |e_i| + 2|e_i|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_i|^j \\
\leq |e_1| + 2|e_1|^{2k+2} + \sum_{j=2k+4}^{4k+3} |e_1|^j = e_1 + 2e_1^{2k+2} + \sum_{j=2k+4}^{4k+3} e_1^j = 1
\]

Thus equality must hold everywhere, but this contradicts (a). It follows that \( e_1 \leq |e_i| \).
Proof of (c)

Since $\hat{A}[e_1(w)] = 0$, for some small neighborhood $\Delta w$ we have $\hat{A}[e_1(w + \Delta w)] = 0$. This gives us:
Since $\hat{A}[e_1(w)] = 0$, for some small neighborhood $\Delta w$ we have $\hat{A}[e_1(w + \Delta w)] = 0$. This gives us:

$$0 = \hat{A}[e_1(w)] - \hat{A}[e_1(w + \Delta w)]$$
Proof of (c)

Since $\hat{A}[e_1(w)] = 0$, for some small neighborhood $\Delta w$ we have $\hat{A}[e_1(w + \Delta w)] = 0$. This gives us:

$$0 = \hat{A}[e_1(w)] - \hat{A}[e_1(w + \Delta w)]$$

$$= [1 - e_1(w) - (w^a + w^b)e_1(w)^{2k+2} - w^{a+b}\sum_{j=2k+4}^{4k+3} e_1(w)^{j}] -$$

$$[1 - e_1(w + \Delta w) - ((w + \Delta w)^a + (w + \Delta w)^b)e_1(w + \Delta w)^{2k+2}$$

$$- (w + \Delta w)^{a+b}\sum_{j=2k+4}^{4k+3} e_1(w + \Delta w)^{j}]$$
Since \( \hat{A}[e_1(w)] = 0 \), for some small neighborhood \( \Delta w \) we have \( \hat{A}[e_1(w + \Delta w)] = 0 \). This gives us:

\[
0 = \hat{A}[e_1(w)] - \hat{A}[e_1(w + \Delta w)]
\]

\[
= [1 - e_1(w) - (w^a + w^b)e_1(w)^{2k+2} - w^{a+b} \sum_{j=2k+1}^{4k+3} e_1(w)^j] - [1 - e_1(w + \Delta w) - ((w + \Delta w)^a + (w + \Delta w)^b)e_1(w + \Delta w)^{2k+2} - (w + \Delta w)^{a+b} \sum_{j=2k+4}^{4k+3} e_1(w + \Delta w)^j]
\]

\[
= e_1(w + \Delta w) - e_1(w) + (w^a + w^b)[e_1(w + \Delta w)^{2k+2} - e_1(w)^{2k+2} + w^{a+b} \sum_{j=2k+4}^{4k+3} [e_1(w + \Delta w)^j - e_1(w)^j]
\]

\[
+ e_1(w + \Delta w)^{2k+2}[(w + \Delta w)^a - w^a + (w + \Delta w)^b - w^b]
\]

\[
+ [\sum_{j=2k+4}^{4k+3} e_1(w + \Delta w)^j][(w + \Delta w)^{a+b} - w^{a+b}]
\]
Proof of (c)

$$0 = \ldots$$
Proof of (c)

\[
0 = \left[ e_1(w + \Delta w) - e_1(w) \right] \left( 1 + (w^a + w^b) \sum_{i=0}^{2k+1} e_1(w + \Delta w)^i e_1(w)^{2k+1-i} \right) \\
+ w^{a+b} \sum_{j=2k+4}^{4k+3} \sum_{i=0}^{j-1} e_1(w + \Delta w)^i e_1(w)^{j-1-i} \\
+ \Delta w \left( e_1(w + \Delta w)^{2k+2} \left[ \sum_{i=0}^{a-1} (w + \Delta w)^i w^{a-1-i} + \sum_{i=0}^{b-1} (w + \Delta w)^i w^{b-1-i} \right] \\
+ \left[ \sum_{j=2k+4}^{4k+3} e_1(w + \Delta w)^j \right] \sum_{i=0}^{a+b-1} (w + \Delta w)^i w^{a+b-1-i} \right)
\]
Proof of (c)

\[
0 = \left[ e_1(w+\Delta w) - e_1(w) \right] \left( 1 + (w^a + w^b) \sum_{i=0}^{2k+1} e_1(w+\Delta w)^i e_1(w)^{2k+1-i} \right.
\]
\[
+ w^{a+b} \sum_{j=2k+4}^{4k+3} \sum_{i=0}^{j-1} e_1(w+\Delta w)^i e_1(w)^{j-1-i} \]
\[
+ \Delta w \left( e_1(w+\Delta w)^{2k+2} \left[ \sum_{i=0}^{a-1} (w+\Delta w)^i w^{a-1-i} + \sum_{i=0}^{b-1} (w+\Delta w)^i w^{b-1-i} \right] \right.
\]
\[
+ \left[ \sum_{j=2k+4}^{4k+3} e_1(w+\Delta w)^j \right] \sum_{i=0}^{a+b-1} (w+\Delta w)^i w^{a+b-1-i} \]
\]

Now, since \( e_i(w) \) is continuous, and \( w^a, w^b, \) and \( w^{a+b} \) are differentiable at \( w = 1 \), we can rearrange terms and take the limit of the above equation, which gives us:

\[
e'_1(w) = \lim_{\Delta w \to 0} \frac{e_1(w+\Delta w) - e_1(w)}{\Delta w}
\]
\[
= \frac{(aw^{a-1} + bw^{b-1})e_1(w)^{2k+2} + (a+b)w^{a+b-1} \sum_{j=2k+4}^{4k+3} e_1(w)^j}{1 + (w^a + w^b)(2k+2)e_1(w)^{2k+1} + wa^b \sum_{j=2k+4}^{4k+3} je_1(w)^{j-1}}
\]

□
We now seek an expression for the coefficients of $z^n$ in our generating function. As before, we will denote this function as:

$$g(w) = \sum_{m>0, l \geq 0} p_{n,m,l} w^{am+bl}$$
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For our calculations, it will be convenient to express the denominator of our generating function in terms of it's partial fraction expansion.

$$\frac{1}{A_w(z)} = \frac{1}{w^{a+b}} \sum_{i=1}^{4k+3} \frac{1}{(z - e_i(w)) \prod_{j \neq i} (e_j(w) - e_i(w))}$$
We rearrange this formula to get:

$$\frac{1}{A_w(z)} = \frac{1}{w^{a+b}} \sum_{i=1}^{4k+3} \frac{1}{(1 - \frac{z}{e_i(w)})} \cdot \frac{1}{e_i(w) \prod_{j \neq i} (e_j(w) - e_i(w))}$$
We rearrange this formula to get:

$$\frac{1}{A_w(z)} = \frac{1}{w^{a+b}} \sum_{i=1}^{4k+3} \frac{1}{1 - \frac{z}{e_i(w)}} \cdot e_i(w) \prod_{j \neq i} (e_j(w) - e_i(w))$$

Key Insight: $\frac{1}{1 - \frac{z}{e_i(w)}}$ represents a geometric series.
Home Stretch

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\]

Key Insight: \(\frac{1}{1 - \frac{z}{e_i(w)}}\) represents a geometric series.

We now sum over all coefficients of \(z^n\) in the numerator of our generating function, which gives us:

\[
g(w) = \sum_{i=1}^{4k+3} \frac{w^{-b}(1 - e_i(w)) + e^{k+2}(w) - e^{2k+2}(w)}{e^n_i(w) \prod_{j \neq i}(e_j(w) - e_i(w))}
\]
If we let $q_i(w)$ denote all terms of $g(w)$ that do not depend on $n$, then $g(w)$ reduces to $\sum_{i=1}^{4k+3} wq_i(w)/e_i^n(w)$. 

Notes (from our Proposition):

Since $e_i^n(w)$ is $\ell$-times differentiable, so is $q_i(w)$. For all $e_i^n(w)$ where $i \geq 2$, we have $e_{n1}(w) \ll e_i^n(w)$ as $n \to \infty$. $g(w) = wq_1(w)/e_{n1}(w) + o(\gamma_n(a,b))$, where $\gamma_n(a,b)$ is dependent only on $a, b$ and is negligible for $n$ large.
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- \( g(w) = wq_1(w)/e_1^n(w) + o(\gamma_a^n, \gamma_b^n) \), where \( \gamma_a^n \) and \( \gamma_b^n \) are dependent only on \( a, b \) and is negligible for \( n \) large.
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\frac{g'(1)}{g(1)} = \frac{q_1'(1)e^n(1) - nq_1(1)e_1^{n-1}(1)e_1'(1)}{e_1^{2n}(1)} \cdot \frac{e_1^n(1)}{q_1(1)}
\]
\[
= \frac{q_1'(1)e^{2n}(1)}{e_1^{2n}(1)q_1(1)} - \frac{nq_1(1)e_1^{2n-1}(1)e_1'(1)}{e_1^{2n}(1)q_1(1)}
\]
\[
= \frac{q_1'(1)}{q_1(1)} - \frac{e_1'(1)}{e_1(1)} n
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We now have everything we need to solve for the mean! Recall that this is equivalent to finding \( g'(1)/g(1) \).

\[
\frac{g'(1)}{g(1)} = \frac{q'_1(1)e^n(1) - nq_1(1)e^{n-1}(1)e'_1(1)}{e^{2n}(1)} \cdot \frac{e^n(1)}{q_1(1)}
\]

\[
= \frac{q'_1(1)e^{2n}(1)}{e^{2n}(1)q_1(1)} - \frac{nq_1(1)e^{2n-1}(1)e'_1(1)}{e^{2n}(1)q_1(1)}
\]

\[
= \frac{q'_1(1)}{q_1(1)} - \frac{e'_1(1)}{e_1(1)}n
\]

Letting \( C_{a,b} = -e'_1(1)/e_1(1) \) and \( d_{a,b} = p'(1)/p_1(1) \) gives us a computable expression for the mean in the form:

\[
C_{a,b}n + d_{a,b} + o(\gamma^n_{a,b})
\]
Since our mean grows linearly with $n$, Miller-Wang [MW] gives us a way to find the variance in the form:

$$h'_{a,b}(1)n + q''_1(1) + o(\tau^n_{a,b})$$

Where $h_{a,b}(w) = \frac{we'_1(w)}{e_1(w)} - C_{a,b}$, and the constants $\tau^n_{a,b} \in (0, 1)$ and $p''_1(1)$ depend only on $a$ and $b$. 

Grand Finale!
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- Using our proposition, it is easy to verify that $C_{a,b}$ and $h'_{a,b} \neq 0$, and thus the mean and variance are not independent of $n$. 

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- Using our proposition, it is easy to verify that $C_{a,b}$ and $h'_{a,b} \neq 0$, and thus the mean and variance are not independent of $n$.
- Now that we have a mean $\mu_n$ and a variance $\sigma_n$ that grow in linear time, we are done!
1. We generalized the Fibonacci recurrence to a broader set of recurrences called the $k$-Skipponaccis.
Recap: What Just Happened?

1. We generalized the Fibonacci recurrence to a broader set of recurrences called the $k$-Skipponaccis.

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1. We generalized the Fibonacci recurrence to a broader set of recurrences called the $k$-Skipponaccis.

2. We proved that every $k$-Skipponacci sequence creates a valid far-difference representation.

3. We proved that the distribution of summands in any $k$-Skipponacci far-difference representation approaches a Gaussian distribution.
We have further generalized the $k$-Skipponacis to recurrence relations of the form $S_n = S_{n-1} + S_{n-x} + S_{n-y}$, where $x$ is the distance between same-sign summands and $y$ is the distance between opposite-sign summands.
Further Research and Open Questions

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Example: The Fibonacci recurrence can be written as:

$$F_n = F_{n-1} + F_{n-2} = F_{n-1} + (F_{n-3} + F_{n-4})$$
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- We believe it can be shown that every far-difference restriction $(x, y)$ uniquely defines a sequence of numbers.

- We want to prove that the number of summands in every $(x, y)$ far-difference representations approaches a Gaussian.
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References


